Gibbs phenomenon

Consider the sign function on $\mathbb{T}$, which is the simplest discontinuous function:

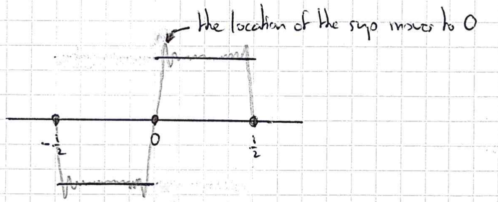
$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } 0 < t < \frac{1}{2} \\ 0 & \text{if } t = 0 \text{ or } \frac{1}{2} \\ -1 & \text{if } -\frac{1}{2} < t < 0 \end{cases}$$

We know from Jordan's criterion (Theorem 2.2) that $S_N \operatorname{sgn}(x) \longrightarrow \operatorname{sgn}(x)$ as $N \to \infty$, for all $x \in \mathbb{T}$. One can also show that $\| S_N \operatorname{sgn} - \operatorname{sgn} \|_p \longrightarrow 0$ for all $1 \leq p < \infty$.

However,

$$\lim_{N\to\infty} \sup_x S_N \operatorname{sgn}(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t}\, dt = 1.17898\ldots$$

Then $\| S_N \operatorname{sgn} - \operatorname{sgn} \|_\infty \not\to 0$. The reason is that $S_N \operatorname{sgn}$ looks like that:



This is known as Gibbs phenomenon, and it is important in image processing.

3. FOURIER SERIES OF $L^p([0,1])$ FUNCTIONS

Let us recall that $L^p([0,1]) \subset L^1([0,1])$ $\forall p \in [1,\infty]$, so that $S_n f$ is well-defined for any $f \in L^p([0,1])$. We consider the two questions

(1) Does $\lim_{n\to\infty} \| S_n f - f \|_p = 0$ for every $f \in L^p([0,1])$?

(2) Does $\lim_{n\to\infty} S_n f(x) = f(x)$ a.e. if $f \in L^p([0,1])$?

Here is a result that helps to answer the first question.

Lemma 3.1

Let $p \in [1,\infty)$. Then $\| S_n f - f \|_p \to 0$ $\forall f \in L^p([0,1])$ iff there exists C_p such that

$$\| S_n f \|_p \leq C_p \, \| f \|_p$$

for all $n \geq 1$, all $f \in L^p([0,1])$.

Proof: We first prove $\Leftarrow$. If g is a trigonometric polynomial, then $S_n g = g$ for all $n \geq \deg g$. Since trigonometric polynomials are dense in L^p, $\forall \varepsilon > 0$ there exists a trig. pol. g such that $\| f - g \|_p < \varepsilon$. Then

$$\begin{aligned} \| S_n f - f \|_p &\leq \| S_n (f-g) \|_p + \| S_n g - g \|_p + \| g - f \|_p \\ &\leq (C_p + 1)\, \varepsilon \end{aligned}$$

provided $n \geq \deg g$. This proves $\Leftarrow$.

The converse claim $\Rightarrow$ follows from the uniform boundedness theorem. (If $\|S_n f - f\|_p \to 0$, then $\|S_n f\|_p < \text{const}$ where the constant depends on f, not on n.) $\square$

According to Lemma 3.1, we have $\|S_n f - f\|_p \to 0$ iff the operator norm of $S_n : L^p([0,1]) \to L^p([0,1])$ is bounded uniformly in n. For $p=1$,

$$\begin{aligned}
\|S_n\| &= \sup_{f\in L^1,\ \|f\|_1=1} \int_0^1 |D_n * f(t)|\, dt \\
&= \sup_f \int_0^1 dt \left| \int_0^1 ds\ D_n(s) f(t-s) \right| \\
&\leq \sup_f \int_0^1 ds\ |D_n(s)| \int_0^1 dt\ |f(t-s)| \\
&= \|D_n\|_1 .
\end{aligned}$$

Furthermore, using $f \approx \delta_0$, we can show that

$$\|S_n\| \geq \int_0^1 dt \left| \int_0^1 ds\ D_n(s)\, \delta_0(t-s) \right| = \int_0^1 dt\ |D_n(t)| = \|D_n\|_1 .$$

We saw in the previous chapter that $\|D_n\| \approx \frac{4}{\pi^2} \log n$. By Lemma 3.1, we can conclude that it is not true that $\|S_n f - f\|_1 \to 0$ $\forall f \in L^1$.

We shall see that it is true for all $1<p<\infty$. And it is not true for $p=\infty$, since the limit of $S_n f$ is continuous if it exists, but there exist discontinuous functions in $L^\infty([0,1])$.

<u>Theorem 3.2</u>

Let $p \in [1,\infty)$. Then

(a) If $f \in L^p([0,1])$, then $\lim_{n\to\infty} \|\sigma_n f - f\|_p = 0$. (Cesàro)

(b) The trigonometric polynomials $\left(\sum_{k=-n}^{n} a_k e^{-2\pi i k x}\right)$ are dense.

(c) If $f \in L^p([0,1])$ and $\hat f(k) = 0$ $\forall k \in \mathbb{Z}$, then $f = 0$.

Let us recall <u>Minkowski inequality</u>: $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, $1 \leq p \leq \infty$, and more generally: If μ, ν are σ-finite measures on Ω, Γ respectively, and f is a non-negative measurable function on $\Omega \times \Gamma$, then

$$\left(\int_\Omega d\mu(x) \left(\int_\Gamma d\nu(y)\, f(x,y) \right)^p \right)^{\frac{1}{p}} \leq \int_\Gamma d\nu(y) \left(\int_\Omega d\mu(x)\, f(x,y)^p \right)^{\frac{1}{p}} .$$

<u>Proof of Theorem 3.2:</u> Recall that $\sigma_n f = F_n * f$ where F_n is the Fejér kernel $F_n(t) = \frac{1}{n} \frac{\sin^2(n\pi t)}{\sin^2 \pi t}$. We have

$$\begin{aligned}
\|\sigma_n f - f\|_p &= \left(\int_{-\frac12}^{\frac12} dt \left| \int_{-\frac12}^{\frac12} ds\ F_n(s)\, [f(t-s) - f(t)] \right|^p \right)^{\frac{1}{p}} \\
&\overset{\text{Minkowski}}{\leq} \int_{-\frac12}^{\frac12} ds\ F_n(s)\, \| f(\cdot - s) - f(\cdot) \|_p \\
&\leq \int_{|s|<\delta} ds\ F_n(s)\, \|f(\cdot-s) - f(\cdot)\|_p + 2\|f\|_p \underbrace{\int_{\delta<|s|<\frac12} F_n(s)\, ds}_{\to 0 \text{ as } n\to\infty}
\end{aligned}$$

For every $\varepsilon > 0$, there exists $g \in C([0,1])$ such that $\|f-g\|_p < \frac{\varepsilon}{3}$, so that

$$\|f(\cdot - s) - f(\cdot)\|_p \leq \|f(\cdot - s) - g(\cdot - s)\|_p + \|g(\cdot - s) - g(\cdot)\|_p + \|g - f\|_p < \varepsilon,$$

provided that δ is small enough. Then $\|\sigma_n - f\|_p$ is arbitrarily small if n is large. This proves (a).

(b) follows since $\sigma_n f$ is a trigonometric polynomial (why? – Write the details!). (c) is also immediate since $\hat{f}(k) \equiv 0$ $\Rightarrow$ $\sigma_n f \equiv 0$ for all n. $\square$

$p=2$ is a special case because $L^2([0,1])$ is a Hilbert space. The functions $(e^{2\pi i k x})_{k \in \mathbb{Z}}$ form an orthonormal set: $\|e^{2\pi i k x}\|_2 = 1$, and, if $k \neq \ell$,

$$(e^{2\pi i k x}, e^{2\pi i \ell x}) = \int_0^1 e^{2\pi i (\ell - k) x} \, dx = \frac{1}{2\pi i (\ell - k)} e^{2\pi i (\ell - k) x} \Big|_0^1 = 0.$$

By Theorem 3.2 (b), the functions $(e^{2\pi i k x})_{k \in \mathbb{Z}}$ form an orthonormal basis. The Fourier coefficient $\hat{f}(k)$ is the projection of f onto $e^{2\pi i k x}$. By the properties of orthonormal bases, we have

$$\|f\|_2^2 = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2.$$

This is <u>Parseval</u> or <u>Plancherel theorem</u>. It implies that the second property of Lemma 3.1 is true with constant $C_2 = 1$.

We should also note that the Fourier series can be seen as a unitary operator from $L^2([0,1])$ to $\ell^2(\mathbb{Z})$.

<u>Theorem 3.3</u>

For every $p \in (1, \infty)$, for every $f \in L^p([0,1])$, we have

$$\|S_n f - f\|_p \to 0$$

as $n \to \infty$.

The proof is rather complicated and is omitted. An option is to consider the "Hilbert transform" $H : L^p \to L^p$ where $\widehat{Hf}(k) = 1_{k \geq 0} \hat{f}(k)$. One can prove that $\|H\| < \infty$ (Riesz), which implies that $C_p < \infty$ in Lemma 3.1.

We conclude the chapter with two applications of Fourier series: the isoperimetric inequality, and Weyl ergodic theorem. In both cases, Fourier theory offers surprising and rather simple proofs.

Isoperimetric inequality ("Dido's problem")

This is a surprising consequence of Fourier series. According to the legend, Princess Elissa of Tyre had to flee her native city, and she arrived at the location of the future Carthage. She asked the Berber king Iarbas for a small piece of land, just as much as could be encompassed by an oxhide. Iarbas agreed, and Elissa cut the oxhide in fine strips so she could encircle an entire hill. Princess Elissa became Queen Dido and the hill became Carthage. This took place around 800 B.C.

The question is what is the maximal area with given length? Or the minimal length that contains a given area? Clearly, the answer is the circle:

Theorem 3.4 (Isoperimetric inequality)

Let Γ be a simple closed curve in $\mathbb{R}^2$ of length ℓ, and $\mathcal{A}$ be the area of the region enclosed by the curve. Then

$$\mathcal{A} \leq \frac{\ell^2}{4\pi},$$

with equality iff Γ is a circle.

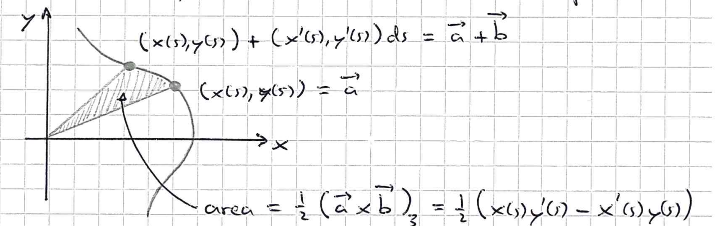
Proof: We can assume that $\ell = 1$ by rescaling. Let $(x(s), y(s))$ be a parametrisation of the curve Γ, with $s \in [0,1]$ and $x'(s)^2 + y'(s)^2 = 1$. Then

$$\int_0^1 \left(x'(s)^2 + y'(s)^2 \right) ds = 1.$$

Since $\widehat{x'}(k) = 2\pi i k \, \hat{x}(k)$, and similarly for y, we have from Plancherel theorem

$$4\pi^2 \sum_{k \in \mathbb{Z}} |k|^2 \left(|\hat{x}(k)|^2 + |\hat{y}(k)|^2 \right) = 1.$$

Next, we need an expression for the area in terms of $(x(s), y(s))$. We could use Green theorem, but we can also derive the expression.



$$\mathcal{A} = \frac{1}{2} \int_0^1 (x(s)y'(s) - x'(s)y(s))\, ds = \pi i \sum_{k \in \mathbb{Z}} k \left(\overline{\hat{x}(k)}\, \hat{y}(k) - \hat{x}(k)\, \overline{\hat{y}(k)} \right)$$

We used Parseval identity, $\int_0^1 \overline{f(s)}\, g(s)\, ds = \sum_k \overline{\hat{f}(k)}\, \hat{g}(k)$, observing that $x(s)$ and $y(s)$ are real. Then