We start with two sufficient conditions on f for $S_N f(x)$ to converge pointwise.

<u>Theorem 2.1</u> (Dini's criterion)

If for some x there exists $\delta > 0$ such that

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty,$$

then $\lim_{N\to\infty} S_N f(x) = f(x)$.

This criterion assumes a bit more than continuity. And indeed, there exist continuous functions for which the Fourier series do not converge pointwise, see below p.

<u>Theorem 2.2</u> (Jordan's criterion)

If f has bounded variation in a neighbourhood of x (i.e. $\exists \delta > 0$ s.t. $\sup_{n, x_1, \ldots, x_n} \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| < \infty$,

where the supremum is over $x_1, \ldots, x_n$ that satisfy $x - \delta = x_0 < x_1 < \ldots < x_n < x_{n+1} = x + \delta$), then

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}\left(f(x+) + f(x-) \right).$$

In order to prove these theorems, we need Riemann-Lebesgue.

<u>Lemma 2.3</u> (Riemann - Lebesgue)

Suppose that $f \in L^1(\mathbb{T})$. Then $\lim_{|k|\to\infty} |\hat{f}(k)| = 0$.

<u>Proof</u>: Since $e^{2\pi i x}$ has period 1,

$$\hat{f}(k) = \int_0^1 f(x)\, e^{-2\pi i k x}\, dx = -\int_0^1 f(x)\, e^{-2\pi i k (x + \frac{1}{2k})}\, dx$$

$$= -\int_0^1 f(x - \tfrac{1}{2k})\, e^{-2\pi i k x}\, dx$$

Then $$\hat{f}(k) = \frac{1}{2} \int_0^1 \left[f(x) - f(x - \tfrac{1}{2k}) \right] e^{-2\pi i k x}\, dx$$

If f is continuous, $\hat{f}(k) \to 0$ by the dominated convergence theorem (check this carefully!). For other functions, $\forall \varepsilon > 0$ $\exists\, g$ continuous such that $\|f - g\|_1 < \frac{\varepsilon}{2}$. Then

$$|\hat{f}(k)| \leq |\widehat{f-g}(k)| + |\hat{g}(k)| \leq \|f - g\|_1 + |\hat{g}(k)| < \varepsilon$$

if $|k|$ is large enough. $\square$

<u>Remark</u>: We have proved a bit more, namely that

$$\lim_{|k|\to\infty} \int_0^1 f(x)\, e^{-2\pi i k x}\, dx = 0$$

whenever $f \in L^1(\mathbb{T})$ and $k \in \mathbb{R}$ (rather than $\mathbb{Z}$). This will be useful.

Proof of Theorem 2.1: We have

$$S_N f(x) - f(x) = \int_{-\frac12}^{\frac12} [f(x-t)-f(x)]\, D_N(t)\, dt = \int_{|t|<\delta} + \int_{\delta<|t|<\frac12}$$

where $D_N(t) = \frac{\sin(\pi(2N+1)t)}{\sin \pi t} = \frac{1}{2i\sin\pi t}\left(e^{i\pi(2N+1)t} - e^{-i\pi(2N+1)t}\right).$

It follows from the assumption that $\frac{f(x-t)-f(x)}{\sin\pi t}\chi_{|t|<\delta}(t)$ is integrable, so the first integral goes to 0 as $N\to\infty$ by Riemann–Lebesgue. The second integral also goes to 0 by R.-L., since $|\sin\pi t|$ is bounded away from 0 for $\delta<|t|<\frac12$. $\square$

Proof of Theorem 2.2: It is obviously harder to prove. Every function of bounded variation can be written as the difference of two monotone functions, so it is enough to prove the claim for f monotone. We have

$$S_N f(x) = \int_{-\frac12}^{\frac12} f(x-t)\, D_N(t)\, dt = \int_0^{\frac12} [f(x-t)+f(x+t)]\, D_N(t)\, dt.$$

We show that

$$\lim_{N\to\infty} \int_0^{\frac12} g(t)\, D_N(t)\, dt = \frac12 g(0+)$$

for all monotone increasing functions g. Theorem 2.2 follows by linearity. We can also assume that $g(0+)=0$. First,

$$\int_0^{\frac12} g(t)\, D_N(t)\, dt = \int_0^{\delta} + \int_\delta^{\frac12}.$$

As in the previous proof, the second integral goes to 0 as $N\to\infty$ by Riemann–Lebesgue. For the first integral, we use the second mean-value theorem for integrals: If ϕ is continuous and h monotone on $[a,b]$, then there exists $c\in(a,b)$ s.t.

$$\int_a^b h\phi = h(b-)\int_c^b \phi + h(a+)\int_a^c \phi.$$

Then there exists $\eta\in(0,\delta)$ such that

$$\int_0^\delta g(t)\, D_N(t)\, dt = g(\delta-)\int_\eta^\delta D_N(t)\, dt.$$

Finally, we check that $\left|\int_\eta^\delta D_N(t)\, dt\right|$ is bounded uniformly in $0<\eta\le\delta<1$ and N. Then the right side above is small by choosing δ small.

$$\left|\int_\eta^\delta D_N(t)\, dt\right| \le \left|\int_\eta^\delta \sin(\pi(2N+1)t)\left(\frac{1}{\sin\pi t}-\frac{1}{\pi t}\right) dt\right| + \left|\int_\eta^\delta \frac{\sin(\pi(2N+1)t)}{\pi t}\, dt\right|$$

$$\le \int_\eta^\delta \left|\frac{1}{\sin\pi t}-\frac{1}{\pi t}\right| dt + 2\sup_{M>0}\left|\int_0^M \frac{\sin\pi t}{t}\, dt\right| < C.$$

$\square$

While continuity is not enough to get the summability of the Fourier series, it is enough for Cesàro summability.

Cesáró and Abel summability

Recall that the series $\sum_{k\geq 0} a_k$ converges if the sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$ converges.

Definition: (a) A series $\sum_{k\geq 0} a_k$ is Cesàro summable if the sequence (σ_n) converges, where

$$\sigma_n = \frac{1}{n}(s_0 + s_1 + \dots + s_n).$$

(b) A series $\sum_{k\geq 0} a_k$ is Abel summable if for any $r \in [0,1)$, the series $A(r) = \sum_{k\geq 0} a_k r^k$ converges, and if $A(r)$ converges as $r \to 1$.

Quite sensibly, the Cesàro sum of $\sum a_k$ is $\sigma = \lim_{n\to\infty} \sigma_n$ and the Abel sum is $A = \lim_{r\to 1} A(r)$. One can check the implications

$$\text{summable} \Rightarrow \text{Cesàro summable} \Rightarrow \text{Abel summable}$$

and that the limits are the same.

Examples: $\sum_{n\geq 0} (-1)^n = \frac{1}{2}$ (Cesàro, but not summable)

$\sum_{n\geq 1} n(-1)^{n+1} = \frac{1}{4}$ (Abel, but not Cesàro)

Theorem 2.4 (Fejér)

Let $f \in L^\infty(\mathbb{T})$. Then

(a) The Fourier series of f is Cesàro summable at every point of continuity of f.

(b) If $f \in C(\mathbb{T})$, its Fourier series is uniformly Cesàro summable.

Proof: We have $\sigma_N f(x) = \frac{1}{N}(S_0 f(x) + \dots + S_N f(x))$ where $S_N f(x) = f * D_N(x)$. Let F_N denote the Fejér kernel such that $\sigma_N f(x) = f * F_N(x)$. We can check that

$$F_N(x) = \frac{1}{N}(D_0(x) + \dots + D_N(x)) = \frac{1}{N}\frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

The kernel satisfies

- $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(x)\,dx = 1 \quad \forall N$
- $\forall \delta > 0, \lim_{N\to\infty} \int_{|x|>\delta} F_N(x)\,dx = 0.$

Heuristically, F_N approaches the Dirac "function" as $N \to \infty$. Let $\varepsilon > 0$ and let δ such that $|f(x-y) - f(x)| < \varepsilon$ for all $|y| < \delta$. Then

$$f * F_N(x) - f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(y)\,[f(x-y) - f(x)]\,dy$$

Then $$|f * F_N(x) - f(x)| \leq \underbrace{\int_{|y|<\delta} F_N(y)\,\varepsilon\,dy}_{\leq \varepsilon} + \underbrace{2\int_{\delta\leq|y|<\frac{1}{2}} |F_N(y)|\,\|f\|_\infty\,dy}_{\to 0 \text{ as } N\to\infty}$$

The right side is as small as we want, by choosing N large and ε small. We have proved that $\sigma_N f(x)$ converges, as $N \to \infty$, to $f(x)$. This is the claim (a).

For (b), recall that if $f \in C(\mathbb{T})$, then f is uniformly continuous since $\mathbb{T}$ is compact. Then δ is uniform in x, and the bound above is also uniform in x. $\square$

Existence of continuous functions with diverging Fourier series

Du Bois-Reymond gave an example of a continuous function whose Fourier series diverges at a dense set of points, in 1873. Here, we use a more abstract argument based on the uniform boundedness theorem of functional analysis.

Let us recall the theorem: If (T_n) are bounded operators $X \to Y$, with X a Banach space and Y a normed space, and $\sup_n \|T_n x\| < \infty$ $\forall x \in X$, then $\sup_n \|T_n\| < \infty$.
The contrapositive is that if (T_n) are bounded operators $X \to Y$ and $\sup_n \|T_n\| = \infty$, then there exists $x \in X$ such that $\sup_n \|T_n x\| = \infty$.

Here, we take $X = C(\mathbb{T})$ with the sup norm, and $Y = \mathbb{R}$.
We let T_n denote the linear functional

$$T_n f = S_n f(0).$$

Let us check that $\|T_n\| < \infty$ in the right sense.

$$\|T_n\| = \sup_{f \in C(\mathbb{T}),\ \|f\|_\infty = 1} |T_n f| = \sup_f \left| \int_0^1 f(t) D_n(t)\, dt \right| = \|D_n\|_1$$

It is clear that $\|D_n\|_1 < \infty$. Further,

$$\begin{aligned}
\|D_n\|_1 &= 2 \int_0^{\frac12} \left| \frac{\sin(\pi(2n+1)t)}{\pi t} \right| dt + O(1) \\
&= \frac{2}{\pi} \int_0^{n+\frac12} \left| \frac{\sin \pi t}{t} \right| dt + O(1) \\
&= \frac{2}{\pi} \sum_{k=0}^{n-1} \int_k^{k+1} \left| \frac{\sin \pi t}{t} \right| dt + O(1) \\
&= \frac{2}{\pi} \sum_{k=0}^{n-1} \int_0^1 \frac{|\sin \pi t|}{t+k}\, dt + O(1) \\
&= \frac{2}{\pi} \int_0^1 |\sin \pi t| \sum_{k=1}^{n-1} \frac{1}{t+k}\, dt + O(1) \\
&= \frac{4}{\pi^2} \log n + O(1).
\end{aligned}$$

Then $\|D_n\|_1 = \|T_n\| \to \infty$, and the uniform boundedness theorem states that there exists a function f such that $|T_n f| \to \infty$, i.e. $|S_n f(0)| \to \infty$.

It is possible to use Baire category theorem and show that there exists a dense set of functions in $C(\mathbb{T})$ whose Fourier series diverges at 0 (or at any other $x \in \mathbb{T}$).