Definition: The characteristic function of a random variable, or Fourier transform, is the function

$$\phi(t) = \mathbb{E}(e^{itX}) = \int_{-\infty}^{\infty} e^{itx}\, dF(x).$$

It exists, and it is continuous by the dominated convergence theorem (once we prove that dF is a measure). It turns out that convergence in distribution is equivalent to pointwise convergence of the Fourier transform.

Theorem 7.2 (Lévy's continuity theorem)
Let X, X_n be random variables with Fourier transforms $\phi(t), \phi_n(t)$. Then

$$X_n \xrightarrow{d} X \quad \text{iff} \quad \phi_n(t) \to \phi(t) \quad \forall t \in \mathbb{R}.$$

Remark: One should not conclude that, if (ϕ_n) converges pointwise, then X_n converges to the r.v. of the limiting characteristic function. Indeed, the limiting function may not be a characteristic function. For instance, consider r.v. whose densities are $g_n(x) = (2\pi n)^{-\frac{1}{2}} e^{-x^2/2n}$. Their characteristic functions are $\phi_n(t) = e^{-\frac{n}{2}t^2}$ and they converge pointwise to 0. The problem can be corrected by requiring continuity of the limiting function at 0.

Proof of Theorem 7.2: We start with $X_n \xrightarrow{d} X \Rightarrow \phi_n(t) \to \phi(t)$ $\forall t$. We have

$$|\phi_n(t) - \phi(t)| = \left| \int_{-\infty}^{-K} e^{itx} (dF_n(x) - dF(x)) + \int_{-K}^{K} \dots + \int_{K}^{\infty} \dots \right|$$

Recall that monotone functions are continuous almost everywhere. For every $\varepsilon > 0$, there exist K, N such that F is continuous at K, and $|F(-K)| < \frac{\varepsilon}{6}$, $|1 - F(K)| < \frac{\varepsilon}{6}$, $|F_n(-K) - F(-K)| < \frac{\varepsilon}{6}$, $|F_n(K) - F(K)| < \frac{\varepsilon}{6}$ for all $n > N$. Then

$$|\phi_n(t) - \phi(t)| < \varepsilon + \left| \int_{-K}^{K} e^{itx} (dF_n(x) - dF(x)) \right|.$$

Next, given $\varepsilon > 0$, let $x_1, \dots, x_M$ be points of continuity of F such that $-K < x_1 < \dots < x_M < K$ and $|x_i - x_{i+1}| < \varepsilon$ $\forall i = 0, \dots, M$ (with $x_0 = -K$ and $x_{M+1} = K$). We have $|e^{itx} - e^{ity}| \leq |x - y| < \varepsilon$ $\forall x, y \in [x_i, x_{i+1}]$ for some i.

Then

$$\int_{-K}^{K} e^{itx} (dF_n(x) - dF(x)) = \sum_{j=0}^{M} \int_{x_j}^{x_{j+1}} e^{itx_j} (dF_n(x) - dF(x)) + \sum_{j=0}^{M} (e^{itx} - e^{itx_j}) (dF_n(x) - dF(x)$$

so that

$$\left| \int_{-K}^{K} e^{itx} (dF_n(x) - dF(x)) \right| \leq \sum_{j=0}^{M} |F_n(x_{j+1}) - F_n(x_j) - F(x_{j+1}) + F(x_j)|$$
$$+ \varepsilon \sum_{j=0}^{M} \left(\int_{x_j}^{x_{j+1}} dF_n(x) + \int_{x_j}^{x_{j+1}} dF(x) \right).$$

The last sum is less than 2ε. The first sum is arbitrarily small by taking n large. So $|\phi_n(t) - \phi(t)| < 4\varepsilon$ for n large enough.

We now prove that $\phi_n(t) \to \phi(t)$ implies $F_n(x) \to F(x)$. Let $g \in S(\mathbb{R})$.
We have

$$\int g(x)\,dF_n(x) \overset{\substack{\text{inverse}\\ \text{Fourier}}}{=} \int dF_n(x) \int dt\, e^{itx}\, \hat{g}(t)$$

$$\overset{\text{Fubini}}{=} \int dt\, \hat{g}(t) \int dF_n(x)\, e^{itx}$$

$$= \int dt\, \hat{g}(t)\, \phi_n(t)$$

$$\longrightarrow \int dt\, \hat{g}(t)\, \phi(t) \quad \text{by dominated conv. } (|\phi_n(t)| \le 1\ \forall t,n)$$

$$= \int dt\, \hat{g}(t) \int dF(x)\, e^{itx}$$

$$\overset{\substack{\text{Fubini}\\ \&\ \text{inverse}\\ \text{Fourier}}}{=} \int g(x)\, dF(x).$$

Theorem 7.2 then follows from the next lemma. □

Lemma 7.3

Let x be a continuity of the distribution function F. Assume that $\int g\,dF_n \to \int g\,dF$ for every $g \in S(\mathbb{R})$. Then $F_n(x) \to F(x)$.

Proof: The counterpositive is: If $F_n(x) \not\to F(x)$, then there exists $g \in S(\mathbb{R})$ such that $\int g\,dF_n \not\to \int g\,dF$. We prove the counterpositive.

If $F_n(x) \not\to F(x)$, then there exists $\delta > 0$ and (n_k) such that $F_{n_k}(x) \to F(x) + \delta$ or $F_{n_k}(x) \to F(x) - \delta$ (since compact sequences have convergent subsequences).

1st case: Let $g \in S(\mathbb{R})$ such that $g(y) = 0$ if $y < x$ or $y > b$ and $g(y) \le 1\ \forall y$. Then

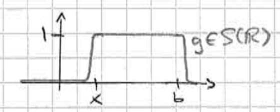
$$\int g\,dF_{n_k}(y) = \lim_{m\to\infty} \sum_{j=1}^{m} \underbrace{g\left(x + \tfrac{b-x}{m} j\right)}_{\le 1} \left[F_{n_k}\left(x + \tfrac{b-x}{m}(j+1)\right) - F_{n_k}\left(x + \tfrac{b-x}{m} j\right) \right]$$

$$\le \underbrace{F_{n_k}(b)}_{\le 1} - \underbrace{F_{n_k}(x)}_{\ge F(x) + \delta - \frac{\delta}{2} \text{ if } n_k \text{ large}} \le 1 - F(x) - \frac{\delta}{2}.$$

This holds $\forall n_k$ large enough. But for every $\delta > 0$, there exists a g as above, such that $\left|\int g\,dF - 1 + F(x)\right| < \frac{\delta}{4}$, which leads to a contradiction.

2nd case: Given $\delta > 0$, let $g \in S(\mathbb{R})$ such that $g(y) = 1$ when $y \in [x,b]$, and $\left|\int g\,dF - 1 + F(x)\right| < \frac{\delta}{2}$.

We have



$$\int g\,dF_{n_k}(y) \ge \lim_{m\to\infty} \sum_{j=1}^{m} g\left(x + \tfrac{b-x}{m} j\right) [\ \ldots\ ]$$

$$\ge \underbrace{F_{n_k}(b)}_{\ge 1 - \frac{\delta}{4} \text{ if } n_k \text{ large, see below}} - F_{n_k}(x) \ge 1 - F(x) + \frac{\delta}{4}$$

Contradiction.

We still need to check that $F_{n_k}(b) \ge 1 - \frac{\delta}{4}$. Let $h \in S(\mathbb{R})$ s.t. $h(y) = 1$ if $y \in [-b, b]$. Then

$$F_{n_k}(b) \ge \int_{-b}^{b} h\,dF_{n_k}(y) \longrightarrow \int_{-b}^{b} h\,dF(y) \ge 1 - \frac{\delta}{2} \text{ if } b \text{ is large enough.}$$

□

Lemma 7.4

Assume that the random variable X has finite second moment, i.e. $\int_{-\infty}^{\infty} x^2\, dF(x) < \infty$. Then its characteristic function is twice differentiable.

Proof: First, notice that $\int_{-\infty}^{\infty} |x|\, dF(x) \leq \int_{-\infty}^{\infty} \frac{1}{2}(1+x^2)\, dF(x) < \infty$.

We have

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$$

$$\phi'(t) = \lim_{h\to 0} \int_{-\infty}^{\infty} \underbrace{\frac{e^{i(t+h)x} - e^{itx}}{h}}_{|\ | \leq |x|}\, dF(x) \underset{\substack{\text{domin.}\\ \text{conv.}}}{=} i \int_{-\infty}^{\infty} e^{itx} x\, dF(x)$$

$$\phi''(t) = \lim_{h\to 0} i \int_{-\infty}^{\infty} \frac{e^{i(t+h)x} - e^{itx}}{h}\, x\, dF(x) \underset{\substack{\text{domin.}\\ \text{conv.}}}{=} -\int_{-\infty}^{\infty} e^{itx} x^2\, dF(x).$$

Finally, we observe that $\phi''(t)$ is continuous by dominated convergence. □

Proof of Theorem 7.1 (the central limit theorem): We use Lévy's continuity theorem, so it is enough to show that $\phi_n(t) \longrightarrow \phi(t)$ for every t, where ϕ_n is the characteristic function of $\frac{X_1 + \dots + X_n - \mu n}{\sigma\sqrt{n}}$ and $\phi(t)$ is the characteristic function of $\mathcal{N}(0,1)$. First, we calculate ϕ. Recall that $\int_{-\infty}^{\infty} e^{-2\pi i k x} e^{-\pi x^2}\, dx = e^{-\pi k^2}$. Then

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \underset{x=\sqrt{2\pi}\, y}{=} \int_{-\infty}^{\infty} e^{i\frac{t}{\sqrt{2\pi}} 2\pi y} e^{-\pi y^2}\, dy = e^{-\frac{t^2}{2}}.$$

Next, we have

$$\begin{aligned} \phi_n(t) &= \mathbb{E}\left(e^{it\frac{X_1 + \dots + X_n - \mu n}{\sigma\sqrt{n}}}\right) \\ &= \prod_{j=1}^{n} \mathbb{E}\left(e^{it\frac{X_j - \mu}{\sigma\sqrt{n}}}\right) \\ &= \left[\phi_{X-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n. \end{aligned}$$

Here, $\phi_{X-\mu}$ is the characteristic function of the random variable $X-\mu$, where X is X_1, or X_2, ... (they are all the same). By Lemma 7.4, we have

$$\phi_{X-\mu}(s) = \underbrace{\phi_{X-\mu}(0)}_{=1} + s\underbrace{\phi'_{X-\mu}(0)}_{=0} + \frac{1}{2}s^2\underbrace{\phi''_{X-\mu}(0)}_{=-\sigma^2} + o(s^2).$$

Then $\left[\phi_n\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n = \left[1 - \frac{1}{2}\frac{t^2}{n} + o\left(\frac{1}{n}\right)\right]^n \longrightarrow e^{-t^2/2}$ as $n \to \infty$. □

8. FOURIER THEORY OF FINITE ABELIAN GROUPS

The heart of Fourier theory is a group structure. In everything we have seen so far, the group of space translations played an important rôle. The goal of this chapter is to show that Fourier theory applies to more general groups, where Fourier functions cannot be written explicitly as before.

We start by reviewing a simple and explicit situation: the group $\mathbb{Z}(N)$. In passing, we look at the "fast Fourier transform", a simple yet important algorithm in applied mathematics. Then we derive Fourier theory for general finite abelian groups.

8.1 The group $\mathbb{Z}(N)$ and $\ell^2(\mathbb{Z}(N))$.

Definition: $\mathbb{Z}(N)$ is the group of all N-th roots of 1.

An element in $\mathbb{Z}(N)$ can be written $e^{2\pi i k/N}$, $k = 0, \dots, N-1$. Since $e^{2\pi i \frac{k}{N}} e^{2\pi i \frac{\ell}{N}} = e^{2\pi i \frac{m}{N}}$ with $m = k + \ell \mod N$, we see that $\mathbb{Z}(N)$ is isomorphic to $\mathbb{Z}/N\mathbb{Z}$.

We now consider the linear space $\ell^2(\mathbb{Z}(N))$ of complex functions on $\mathbb{Z}(N)$, with the natural inner product: if $f, g \in \ell^2(\mathbb{Z}(N))$,

$$(f,g) = \sum_{k=0}^{N-1} \overline{f(k)}\, g(k).$$

Let e_m, $m = 0, \dots, N-1$, be the following functions: $e_m(k) = e^{2\pi i \frac{mk}{N}}$.

Lemma 8.1

The set $\{e_m\}$ is orthogonal, $(e_\ell, e_m) = \begin{cases} N & \text{if } \ell = m \\ 0 & \text{if } \ell \neq m \end{cases}$.

Proof: $(e_\ell, e_m) = \sum_{k=0}^{N-1} \overline{e_\ell(k)}\, e_m(k) = \sum_{k=0}^{N-1} e^{2\pi i \frac{(m-\ell)k}{N}}$.

The result is clear for $\ell = m$. For $\ell \neq m$, we have $e^{2\pi i \frac{\ell - m}{N}} \neq 1$, so that

$$\sum_{k=0}^{N-1} e^{2\pi i \frac{m-\ell}{N} k} = \frac{1 - e^{2\pi i (m-\ell)}}{1 - e^{2\pi i \frac{m-\ell}{N}}} = 0. \qquad \square$$

Remark: As we will see later, the functions e_m are the characters of $\mathbb{Z}(N)$, and the lemma can be proved in much greater generality.

Definition: The n-th Fourier coefficient a_n of f is

$$a_n = \frac{1}{N} \sum_{k=0}^{N-1} f(k)\, e^{-2\pi i \frac{kn}{N}}.$$

Theorem 8.2 (Parseval–Plancherel)

For any $f \in \ell^2(\mathbb{Z}(N))$, we have $f(k) = \sum_{n=0}^{N-1} a_n\, e^{2\pi i \frac{nk}{N}}$ and $\sum_{n=0}^{N-1} |a_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |f(k)|^2$.