Term 1, 2013–14

FOURIER ANALYSIS (MA 433)

Lectures: Monday 13–14, MS.03
Thursday 12–13, B3.03
Friday 10–11, B3.02

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More information on the web page: http://www.ueltschi.org/teaching.html

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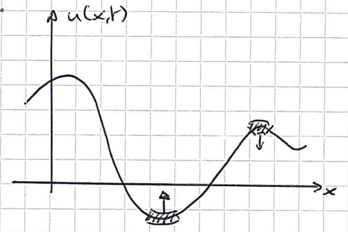
1. INTRODUCTION

Let us start with the description of a string. It is represented by the function $u(x,t)$, that gives the height at position x and time t.



One can argue that the forces in the string are related to the curvature, so that Newton's law leads to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The derivation of this equation is actually a fine piece of modelisation, everybody should look it up!

Solution using traveling waves

This is the simpler method, it should not be overlooked.

Change of variables $\xi = x+t$, $\eta = x-t$, $v(\xi,\eta) = u(x,t)$.

The wave equation becomes $\dfrac{\partial^2 v}{\partial \xi \partial \eta} = 0.$

Integrating, we get $v(\xi,\eta) = F(\xi) + G(\eta)$, so that

$$u(x,t) = F(x+t) + G(x-t)$$

For some functions F, G. u is the superposition of two travelling waves (explain why!).

One is often interested in finding the evolution of the string given the initial conditions

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x),$$

For some given functions f, g. The solution is given by d'Alembert formula

$$u(x,t) = \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy.$$

The equations above apply to an infinite string, but they can be extended to a finite string using the following trick. Suppose that the string is attached at $x=0$ and $x=\pi$. Then $f(0) = f(\pi) = 0$ and $g(0) = g(\pi) = 0$. f, g are functions on $[0,\pi]$, but we extend them on $\mathbb{R}$ by making them odd on $[-\pi,\pi]$, then periodic of period 2π.

d'Alembert formula gives a solution $u(x,t)$ that satisfies $u(0,t) = u(\pi,t) = 0$ and $\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(\pi,t) = 0$.

(Check it carefully!)

Solution using standing waves

This method is more complicated, but it is more revealing and more general. It is the origin of Fourier theory.

We look for solutions of the form $u(x,t) = \varphi(x)\,\psi(t)$ ("separation of variables"). The wave equation gives

$$\varphi(x)\,\psi''(t) = \varphi''(x)\,\psi(t) \iff \frac{\psi''(t)}{\psi(t)} = \frac{\varphi''(x)}{\varphi(x)}$$

It follows that, for some λ,

$$\begin{cases} \psi''(t) - \lambda\psi(t) = 0 \\ \varphi''(x) - \lambda\varphi(x) = 0 \end{cases}$$

If we look for solutions that oscillate, we need $\lambda < 0$.
Write $\lambda = -m^2$. Solutions are of the form

$$\psi(t) = A\cos mt + B\sin mt$$
$$\varphi(x) = \tilde{A}\cos mx + \tilde{B}\sin mx$$

If $u(0,t) = u(\pi,t) = 0$ and $\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(\pi,t) = 0$, we must have $\tilde{A} = 0$ and $m \in \mathbb{Z}$. It is enough to consider $m \in \mathbb{N}$ (redefining constants if necessary). We get the solution

$$u_m(x,t) = (A_m\cos mt + B_m\sin mt)\sin mx$$

This is a standing wave! By superposition, the following is solution:

$$u(x,t) = \sum_{m=1}^{\infty}(A_m\cos mt + B_m\sin mt)\sin mx$$

If f is the initial condition, we have $\sum_{m=1}^{\infty} A_m \sin mx = f(x)$.
This leads to a natural question: can we find (A_m) so that the series above gives f? Here is a useful calculation:

$$\int_0^\pi f(x)\sin nx\,dx = \int_0^\pi \sum_{m=1}^{\infty} A_m \sin mx \sin nx\,dx =$$
$$= \sum_{m=1}^{\infty} A_m \underbrace{\int_0^\pi \sin mx \sin nx\,dx}_{\frac{\pi}{2}\delta_{m,n}} = \frac{\pi}{2}A_n$$

Then if the series (A_m) exists, and it converges fast enough so the exchange of sum and integral above is justified, we must have $A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$.
We can consider the interval $[-\pi,\pi]$ instead of $[0,\pi]$, $m \in \mathbb{Z}$ instead of $\mathbb{N}$, and the functions e^{imx} instead of $\sin nx$.
The question becomes: Letting $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$, is it true that $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$?

At the time of Fourier (1768 – 1830), this question was fascinating and confusing. The notion of convergence was not clarified yet. Claims were often imprecise and easily refuted. Many prominent mathematicians (including Euler, d'Alembert, ...) believed the answer to the question above to be negative.

An interesting observation is that if $\sum_{n=-N}^{N} a_n e^{inx}$ converges as $N \to \infty$ to a function f with respect to the $\|\cdot\|_\infty$ norm, then f is necessarily continuous. This follows from the fact that $C([-\pi,\pi], \|\cdot\|_\infty)$ is a complete space. It follows that if we want to give a positive answer for discontinuous functions, we must consider other topologies.

2. FOURIER SERIES

We now consider the Fourier series of integrable functions, and we look for criteria for pointwise convergence. Let us be more precise. Let $f \in L^1(\mathbb{T})$, where $\mathbb{T}$ is the torus $[0,1]$ with periodic boundary conditions (equivalently, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$). The Fourier coefficients are defined by

$$\hat{f}(k) = \int_0^1 f(x)\, e^{-2\pi i k x}\, dx, \qquad k \in \mathbb{Z}.$$

Let $S_N f$ denote the partial sum of the Fourier series:

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k)\, e^{2\pi i k x}$$

This partial sum is conveniently written with the help of the Dirichlet kernel $D_N(t)$:

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi i k t} = \frac{\sin(\pi(2N+1))}{\sin \pi t}$$

This kernel satisfies (check this!)

$$\int_0^1 D_N(t)\, dt = 1 \qquad \text{and} \qquad |D_N(t)| \le \frac{1}{\sin \pi\delta}, \quad \delta \le |t| \le \tfrac{1}{2}$$

Then $S_N f(x)$ can be written as

$$S_N f(x) = \int_0^1 f(x-t)\, D_N(t)\, dt = f * D_N(x)$$

Heuristically, D_N tends to the Dirac function at 0, but the precise meaning of the convergence is not immediate.