Assignment 4

1. Truncating $L^p(\mathbb{R}^d)$ functions.

- Let $f \in L^2(\mathbb{R}^d)$, and define $f_n(x) = f(x)\,\chi_{\|x\|<n}$. Show that $f_n \in L^1(\mathbb{R}^d)$ and that
$$\lim_{n\to\infty} \|f_n - f\|_2 = 0.$$
- Let $f \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq 2$. Let $f_1(x) = f(x)\,\chi_{|f|\geq 1}(x)$ and $f_2(x) = f(x)\,\chi_{|f|<1}(x)$. Show that $f_1 \in L^1$ and $f_2 \in L^2$.

2. Consistency of the definition of the L^2 Fourier transform.

- Suppose that (f_n) and (g_n) are sequences in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ that both converge to $f \in L^2$. Define $\widehat{f}_n, \widehat{g}_n$ using the L^1 transform. Show that $(\widehat{f}_n)$ and $(\widehat{g}_n)$ converge to the same limit. (This limit is $\widehat{f}$ by definition).
- Extend Plancherel's identity from functions in $L^1 \cap L^2$, to any function in L^2.

3. Show that the Hermite functions,
$$h_n(x) = \frac{(-1)^n}{n!}\,\mathrm{e}^{-\pi x^2}\,\frac{\mathrm{d^n}}{\mathrm{dx^n}}\,\mathrm{e}^{-2\pi x^2},$$
are eigenvectors of the $L^2(\mathbb{R})$ Fourier transform.

Hint: Compute the generating function
$$\sum_{n\geq 0} t^n h_n(x).$$
This gives a Gaussian, and one can get its Fourier transform. Equating the powers of t, one should obtain the result.

4. Show that the Hausdorff-Young inequality is false for $p > 2$.

Hint: Extend Proposition 4.1 (Fourier transform of a Gaussian) to Gaussians with complex parameters λ such that $\operatorname{Re}\lambda > 0$. Then consider the Gaussian g_λ with $\lambda = (a + \mathrm{i}b)^{-1}$, $a > 0$, and show that $\|g_\lambda\|_p \sim |b|^{1/2}$ and $\|\widehat{g}_\lambda\|_q \sim |b|^{1/q}$ as $|b| \to \infty$.