Assignment 8

1. Fast Fourier transform. Show that, for $N = 3^n$, the coefficients a_k^N can be computed with less than $6N \log_3 N$ operations. What about $N = 4^n, 5^n \ldots$?

2. Let e be a character on the group $\mathbb{Z}(N)$. Show that there exists a unique $0 \leq \ell \leq N-1$ such that

$$e(k) = \mathrm{e}^{2\pi \mathrm{i} \ell \mathrm{k}/\mathrm{N}}$$

for all $k \in \mathbb{Z}(N)$. Conversely, every such function is a character. (*Hint:* Show that $e(1)$ is the Nth root of 1.)

3. Infinite abelian groups S^1 and $\mathbb{R}$. A character here is a *continuous* function e that satisfies $|e(x)| = 1$ and $e(x+y) = e(x)e(y)$.

(a) Prove that all characters on S^1 are given by

$$e_n(x) = \mathrm{e}^{2\pi \mathrm{inx}}$$

with $n \in \mathbb{Z}$.

(*Hint:* If f is continuous and $f(x+y) = f(x)f(y)$, then f is differentiable. To see this, note that if $f(0) \neq 0$, then for small δ, $c = \int_0^\delta f(y)\mathrm{dy} \neq 0$ and $cf(x) = \int_x^{x+\delta} f(y)\mathrm{dy}$. Differentiate to conclude that $f(x) = \mathrm{e}^{Ax}$ for some A.)

(b) Prove that all characters on $\mathbb{R}$ are of the form

$$e_k(x) = \mathrm{e}^{2\pi \mathrm{ikx}}$$

with $k \in \mathbb{R}$.

4. Let G be a finite abelian group and $e : G \to \mathbb{C}$ a function that satisfies $e(a \cdot b) = e(a)e(b)$ for all $a, b \in G$. Prove that e is either identically zero, or e never vanishes. In the second case, show that, for each a, $e(a) = \mathrm{e}^{2\pi \mathrm{ir}}$ for some rational r of the form $r = p/|G|$, $p \in \mathbb{N}$.

5. The convolution of two functions $f, g \in \ell^2(G)$ where G is is a finite abelian group, is defined by

$$(f * g)(a) = \frac{1}{|G|} \sum_{b \in G} f(b) g(a \cdot b^{-1}).$$

Show that

$$\widehat{(f * g)}(e) = \widehat{f}(e) \widehat{g}(e)$$

for all characters $e \in \widehat{G}$.