Assignment 7

1. Find the Fourier transform of $\mathrm{PV}\frac{1}{x}$. You may use the results of previous exercises.

2. Show that $\phi_n \to \phi$ implies $\widehat{\phi}_n \to \widehat{\phi}$, where convergence is in the sense of the Schwartz space.

3. Give a full proof of Proposition 6.4: For all multi-indices α,

(a) $\partial^\alpha \widehat{T} = (-2\pi \mathrm{i})^\alpha \, \widehat{\mathrm{x}^\alpha \mathrm{T}}$.

(b) $\widehat{\partial^\alpha T} = (2\pi \mathrm{i}\, \mathrm{k})^\alpha \, \widehat{\mathrm{T}}$.

4. Consider the function $f(x) = \log|x|$ on $\mathbb{R}$.

(a) Show that f is locally integrable. In case you need the primitive of $\log x$, try to differentiate $x \log x$, that may help!

(b) Show that the distributional derivative of $\log|x|$ is $\mathrm{PV}\frac{1}{x}$.

5. Consider the *Airy equation* in one dimension

$$u'' - xu = 0.$$

A solution is the *Airy function* $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}t^3 + xt)\mathrm{dt}$. It is named after the British George Biddell Airy who became Astronomer Royal in 1835.

(a) Check formally that $\mathrm{Ai}(x)$ is solution to the Airy equation. Hint: differentiate under the integral sign; the primitive of $(t^2 + x)\cos(\frac{1}{3}t^3 + xt)$ is easily found.

(b) Let

$$u(x) = c\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mathrm{k}^3/3 + \mathrm{i}\mathrm{k}\mathrm{x}} \mathrm{dk}.$$

Give u a rigorous meaning as the Fourier transform of a distribution. Prove that the distribution u satisfies the Airy equation. Hint: take the Fourier transform of the equation.