Assignment 6

1. Suppose that $f \not\equiv 0$ is a bounded function on $\mathbb{R}$ with compact support. Prove that its Fourier transform does not have compact support.

Hint: Show that $|\frac{\mathrm{d}^n}{\mathrm{d}k^n}\widehat{f}(k)| < C^n$ for some constant C, so that $\widehat{f}$ has a Taylor series around 0 with infinite radius of convergence. Then $\widehat{f}$ is analytic and cannot have compact support.

2. Let $d = 3$. Find the Fourier transform of the function $1/\|x\|$. Since this function is not in any L^p space, explain the meaning of the Fourier transform.

Hint: Think about distributions, and remember that the Fourier transform of the function $\frac{1}{|x|}\,\mathrm{e}^{-2\pi\mu|x|}$ is $\frac{1}{\pi}\frac{1}{k^2+\mu^2}$.

3. Prove that $\frac{1}{x+\mathrm{i}0} = \mathrm{PV}\frac{1}{x} - \mathrm{i}\pi\delta_0$. Hint: use $\frac{1}{x+\mathrm{i}\varepsilon} = \frac{x}{x^2+\varepsilon^2} - \frac{\mathrm{i}\varepsilon}{x^2+\varepsilon^2}$.

4. Compute rigorously the Fourier transforms of

(a) δ_{x_0}.

(b) x.

(c) $x\delta_0$.

5. Compute rigorously the Fourier transform of $\frac{1}{x+\mathrm{i}0}$. Hint: Obtain the following expression:

$$\widehat{\frac{1}{x+\mathrm{i}0}}(\phi) = \lim_{\varepsilon\searrow 0}\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{x+\mathrm{i}\varepsilon}\left[\int_{-\infty}^{\infty}\mathrm{e}^{-2\pi\mathrm{i}kx}\phi(k)\,\mathrm{d}k\right]\mathrm{d}x.$$

Then use Fubini and contour methods.