Assignment 5

1. Prove that the Fourier transform of a Schwartz function, $f \in \mathcal{S}(\mathbb{R}^d)$, is a Schwartz function.

2. Area and volume of the unit sphere. You can skip this exercise if you know it already, but it is otherwise a must-do!

Let $|S_d|$ be the area of the d-dimensional unit sphere (that is, the surface is d-dimensional), and $|B_d|$ the volume of the d-dimensional ball.

(a) Show that

$$|S_d| = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})},$$

where $\Gamma(s) = \int_0^\infty \mathrm{e}^{-t} t^{s-1} \mathrm{dt}$ is the Gamma function.

Hint: Calculate $\int_{\mathbb{R}^d} \mathrm{e}^{-\pi \|x\|^2} \mathrm{dx} = 1$ using polar coordinates.

(b) Show that $d|B_d| = |S_{d-1}|$, hence

$$|B_d| = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}.$$

3. Heat equation, formal calculations. Consider the heat equation in $\mathbb{R}^d$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_d^2},$$

for $t > 0$, with initial value $u(x, 0) = f(x)$ and $f \in \mathcal{S}(\mathbb{R}^d)$.

Take the Fourier transform of the heat equation, and solve it for any fixed k. Show that the formal solution can be written

$$u(x,t) = \int_{\mathbb{R}^d} \widehat{f}(k)\, \mathrm{e}^{-4\pi^2 t \|k\|^2}\, \mathrm{e}^{2\pi \mathrm{ikx}} \mathrm{dk}.$$

4. Heat equation, rigorously. Define the **heat kernel** $H_t(x)$ by

$$H_t(x) = \frac{1}{(4\pi t)^{d/2}}\, \mathrm{e}^{-\|x\|^2/4t} = \int_{\mathbb{R}^d} \mathrm{e}^{-4\pi^2 t \|k\|^2}\, \mathrm{e}^{2\pi \mathrm{ikx}} \mathrm{dk}.$$

We consider the function $u(x,t) = (f * H_t)(x)$.

(a) Show that $u(x,t)$ is equal to the formal solution of Exercise 3.

(b) Show that $u(x,t)$ is a Schwartz function in x, for each fixed $t > 0$.

(c) Show that $u(x,t)$ solves the heat equation for each $t > 0$.

(d) Show that $\lim_{t\to 0+} u(x,t) = f(x)$, uniformly in x.