Assignment 3

1. Show that $\int_0^\infty \frac{\sin x}{x} \mathrm{dx} = \frac{\pi}{2}$.
Hint: Use $\int_0^\infty \frac{\sin x}{x} \mathrm{dx} = \lim_{\mathrm{n}\to\infty} \int_0^1 \frac{\sin(\mathrm{nx})}{\mathrm{x}} \mathrm{dx}$. This is almost the integral of the Dirichlet kernel, and we know that $\int_0^1 D_n(x) \mathrm{dx} = 1$. Use Riemann-Lebesgue to show that the difference goes to 0.

2. Properties of the Fourier transform. Suppose that $f \in L^1(\mathbb{R})$. Prove the following facts:

(a) **Riemann-Lebesgue lemma**.

$$\lim_{|k|\to\infty} \widehat{f}(k) = 0.$$

(b) Convolutions: $\widehat{f * g}(k) = \widehat{f}(k)\widehat{g}(k)$.

(c) If $f(x)$ and $xf(x)$ are functions in $L^1(\mathbb{R}^d)$, show that $\widehat{f}$ is differentiable, and that

$$\frac{\mathrm{d}}{\mathrm{dk}} \widehat{f}(k) = -2\pi \mathrm{i} \widehat{\mathrm{xf}}(\mathrm{k}).$$

(d) If f and $\frac{\mathrm{d}}{\mathrm{dx}} f$ are functions in $L^1(\mathbb{R}^d)$, show that

$$\widehat{\frac{\mathrm{d}}{\mathrm{dx}}}(k) = 2\pi \mathrm{ik} \widehat{\mathrm{f}}(\mathrm{k}).$$

3. Let $d = 1$. Show that the Fourier transform of $\frac{1}{x^2+\mu^2}$ is $\frac{\pi}{\mu} \mathrm{e}^{-2\pi\mu|k|}$ (use contour methods).

4. Let $d = 1$. Show that the Fourier transform of $\mathrm{e}^{-2\pi\mu|x|}$ is $\frac{\mu}{\pi} \frac{1}{k^2+\mu^2}$.

5. Show that if $f \in L^1(\mathbb{R}^3)$ is invariant under rotations, then $\widehat{f}$ is also invariant under rotations.

6. Let $d = 3$. Show that the Fourier transform of $\frac{1}{|x|} \mathrm{e}^{-2\pi\mu|x|}$ is $\frac{1}{\pi} \frac{1}{k^2+\mu^2}$.