Assignment 2

1. Prove that the Fejér kernel "converges to Dirac". Precisely, show that $F_n(t) = \frac{1}{n} \frac{\sin^2(n\pi t)}{\sin^2(\pi t)}$ satisfies

(a) $F_n(t) \geq 0$. (Obvious!)

(b) $\int_{-\frac{1}{2}}^{\frac{1}{2}} F_n(t) \mathrm{d}t = 1$.

(c) $\lim_{n\to\infty} \int_{|t|>\delta} F_n(t) \mathrm{d}t = 0$ for any $\delta > 0$.

2. Properties of convolution. Suppose that $f, g, h \in L^1([0,1])$. Prove the following facts:

(a) $f * (g + h) = f * g + f * h$.

(b) $(cf) * g = c(f * g) = f * (cg)$ for any $c \in \mathbb{C}$.

(c) $f * g = g * f$.

(d) $(f * g) * h = f * (g * h)$.

(e) $f * g$ is continuous.

(f) $\widehat{f * g}(k) = \widehat{f}(k)\, \widehat{g}(k)$.

3. Abel summability. Let $f \in L^1([0,1])$, and with $r \in (0,1)$, define

$$A_r f(x) = \sum_{k\in\mathbb{Z}} r^{|k|} \widehat{f}(k)\, \mathrm{e}^{2\pi \mathrm{i} k x}.$$

(a) Prove that this series converges absolutely for all $x \in [0,1]$, and that $A_r f(x)$ is a continuous function of x.

(b) Let P_r be the Poisson kernel such that $A_r f = P_r * f$. Show that

$$P_r(x) = \sum_{k\in\mathbb{Z}} r^{|k|}\, \mathrm{e}^{2\pi \mathrm{i} k x} = \frac{1 - r^2}{1 - 2r\cos(2\pi x) + r^2}.$$

(c) Prove that if $f \in L^p([0,1])$, $1 \leq p < \infty$, we have $\lim_{r \nearrow 1} \|A_r f - f\|_p = 0$. (This should be similar to the proof of Theorem 3.2 (a).) What about $p = \infty$?

4. Show that the Fourier series of a periodic differentiable function $f \in C^1(\mathbb{T})$ is absolutely convergent. (*Hint:* Use the Cauchy-Schwarz inequality and Parseval formula for f'.)

5. Let $f \in C^\alpha(\mathbb{T})$, with $\alpha \in \mathbb{N}$. Show that $\widehat{f}(k) = o(1/|k|^\alpha)$.