Assignment 1

1. Relations between symmetries and Fourier coefficients. Let $f \in L^1(\mathbb{T})$.

(a) Assuming that the Fourier series converges absolutely, show that

$$f(x) = \widehat{f}(0)+\sum_{n\geq 1}\big[\widehat{f}(n)+\widehat{f}(-n)\big]\cos(2\pi nx)+i\big[\widehat{f}(n)-\widehat{f}(-n)\big]\sin(2\pi nx).$$

(b) Show that $\widehat{f}$ is even if f is even (that is, $\widehat{f}(k) = \widehat{f}(-k)$). Then f is given by a cosine series.

(c) Show that $\widehat{f}$ is odd if f is odd. Then f is given by a sine series.

(d) Suppose that $f(x+\frac{1}{2}) = f(x)$ for all $x \in \mathbb{T}$. Show that $\widehat{f}(k) = 0$ for odd k.

(d) Show that f is real-valued if and only if $\overline{\widehat{f}(k)} = \widehat{f}(-k)$.

2. Let f be the odd function such that $f(x) = x(\frac{1}{2}-x)$ on $[0,\frac{1}{2}]$. Draw the graph of the function, and prove that

$$f(x) = \frac{8}{\pi}\sum_{n\geq 1,\,\text{odd}}\frac{\sin(2\pi nx)}{n^3}.$$

3. (a) Let $f(x) = |x|$ on $[-\frac{1}{2},\frac{1}{2}]$. Show that

$$\widehat{f}(k) = \begin{cases} 1/4 & \text{if } k=0,\\ \frac{(-1)^k-1}{2\pi^2k^2} & \text{if } k\neq 0.\end{cases}$$

(b) Let $f(x) = x^2$ on $[-\frac{1}{2},\frac{1}{2}]$. Show that

$$\widehat{f}(k) = \begin{cases} 1/12 & \text{if } k=0,\\ \frac{\cos(\pi k)}{2\pi^2k^2} & \text{if } k\neq 0.\end{cases}$$

(c) Prove the following relations. (The results above may help!)

$$\sum_{n\geq 1,\,\text{odd}}\frac{1}{n^2} = \frac{\pi^2}{8}, \qquad \sum_{n\geq 1}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

4.(a) Let $(a_n)_{n\geq 1}$ be a sequence of real numbers that decrease monotonically to 0, and $(b_n)_{n\geq 1}$ be a sequence of complex numbers such that all partial sums are bounded. Prove that $\sum_n a_n b_n$ converges. (This is **Dirichlet's test** for the convergence of series). You may want to prove and use the following identity:

$$\sum_{n=M}^{N} a_n b_n = a_N \sum_{n=1}^{N} b_n - a_M \sum_{n=1}^{M-1} b_n - \sum_{n=M}^{N-1} (a_{n+1} - a_n) \sum_{m=1}^{n} b_m.$$

(This is a discrete summation by parts.)

(b) Check that the Fourier series with coefficients $\widehat{f}(k) = 1/k$, for $k \neq 0$, converges.

(c) Find the function f, and plot it.