Define the **Landau** kernels by

$$L_n(x) = \begin{cases} \dfrac{(1-x^2)^n}{c_n} & \text{if } -1 \le x \le 1, \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

where c_n is chosen so that $\int_{-\infty}^{\infty} L_n(x)\, dx = 1$. Prove that $\{L_n\}_{n\ge 0}$ is a family of good kernels as $n \to \infty$. As a result, show that if f is a continuous function supported in $[-1/2, 1/2]$, then $(f * L_n)(x)$ is a sequence of polynomials on $[-1/2, 1/2]$ which converges uniformly to f.

[Hint: First show that $c_n \ge 2/(n+1)$.]

11. Suppose that u is the solution to the heat equation given by $u = f * \mathcal{H}_t$ where $f \in \mathcal{S}(\mathbb{R})$. If we also set $u(x, 0) = f(x)$, prove that u is continuous on the closure of the upper half-plane, and vanishes at infinity, that is,

$$u(x,t) \to 0 \qquad \text{as } |x| + t \to \infty.$$

[Hint: To prove that u vanishes at infinity, show that (i) $|u(x,t)| \le C/\sqrt{t}$ and (ii) $|u(x,t)| \le C/(1+|x|^2) + Ct^{-1/2}e^{-cx^2/t}$. Use (i) when $|x| \le t$, and (ii) otherwise.]

12. Show that the function defined by

$$u(x,t) = \frac{x}{t}\mathcal{H}_t(x)$$

satisfies the heat equation for $t > 0$ and $\lim_{t\to 0} u(x,t) = 0$ for every x, but u is *not* continuous at the origin.

[Hint: Approach the origin with (x,t) on the parabola $x^2/4t = c$ where c is a constant.]

13. Prove the following uniqueness theorem for harmonic functions in the strip $\{(x, y) : 0 < y < 1,\ -\infty < x < \infty\}$: if u is harmonic in the strip, continuous on its closure with $u(x, 0) = u(x, 1) = 0$ for all $x \in \mathbb{R}$, and u vanishes at infinity, then $u = 0$.

14. Prove that the periodization of the Fejér kernel $\mathcal{F}_N$ on the real line (Exercise 9) is equal to the Fejér kernel for periodic functions of period 1. In other words,

$$\sum_{n=-\infty}^{\infty} \mathcal{F}_N(x+n) = F_N(x),$$

when $N \ge 1$ is an integer, and where

$$F_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \frac{1}{N}\frac{\sin^2(N\pi x)}{\sin^2(\pi x)}.$$

15. This exercise provides another example of periodization.

(a) Apply the Poisson summation formula to the function g in Exercise 2 to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}$$

whenever α is real, but not equal to an integer.

(b) Prove as a consequence that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)} = \frac{\pi}{\tan \pi\alpha} \tag{15}$$

whenever α is real but not equal to an integer. [Hint: First prove it when $0 < \alpha < 1$. To do so, integrate the formula in (b). What is the precise meaning of the series on the left-hand side of (15)? Evaluate at $\alpha = 1/2$.]

16. The Dirichlet kernel on the real line is defined by

$$\int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi = (f * \mathcal{D}_R)(x) \quad \text{so that} \quad \mathcal{D}_R(x) = \widehat{\chi_{[-R,R]}}(x) = \frac{\sin(2\pi R x)}{\pi x}.$$

Also, the modified Dirichlet kernel for periodic functions of period 1 is defined by

$$D_N^*(x) = \sum_{|n|\le N-1} e^{2\pi i n x} + \frac{1}{2}(e^{-2\pi i N x} + e^{2\pi i N x}).$$

Show that the result in Exercise 15 gives

$$\sum_{n=-\infty}^{\infty} \mathcal{D}_N(x+n) = D_N^*(x),$$

where $N \ge 1$ is an integer, and the infinite series must be summed symmetrically. In other words, the periodization of $\mathcal{D}_N$ is the modified Dirichlet kernel D_N^*.

17. The **gamma function** is defined for $s > 0$ by

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\, dx.$$

(a) Show that for $s > 0$ the above integral makes sense, that is, that the following two limits exist:

$$\lim_{\substack{\delta \to 0 \\ \delta > 0}} \int_{\delta}^{1} e^{-x} x^{s-1}\, dx \quad \text{and} \quad \lim_{A\to\infty} \int_1^A e^{-x} x^{s-1}\, dx.$$

(b) Prove that $\Gamma(s+1) = s\Gamma(s)$ whenever $s > 0$, and conclude that for every integer $n \geq 1$ we have $\Gamma(n+1) = n!$.

(c) Show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \text{and} \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

[Hint: For (c), use $\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$.]

18. The **zeta function** is defined for $s > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$. Verify the identity

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^{\infty} t^{\frac{s}{2}-1}(\vartheta(t) - 1)\, dt \qquad \text{whenever } s > 1$$

where Γ and ϑ are the gamma and theta functions, respectively:

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}\, dt \quad \text{and} \quad \vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}.$$

More about the zeta function and its relation to the prime number theorem can be found in Book II.

19. The following is a variant of the calculation of $\zeta(2m) = \sum_{n=1}^{\infty} 1/n^{2m}$ found in Problem 4, Chapter 3.

(a) Apply the Poisson summation formula to $f(x) = t/(\pi(x^2 + t^2))$ and $\hat{f}(\xi) = e^{-2\pi t|\xi|}$ where $t > 0$ in order to get

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{t}{t^2 + n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi t|n|}.$$

(b) Prove the following identity valid for $0 < t < 1$:

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{t}{t^2 + n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}$$

as well as

$$\sum_{n=-\infty}^{\infty} e^{-2\pi t|n|} = \frac{2}{1 - e^{-2\pi t}} - 1.$$

(c) Use the fact that

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} z^{2m},$$

where B_k are the Bernoulli numbers to deduce from the above formula,

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

20. The following results are relevant in information theory when one tries to recover a signal from its samples.

Suppose f is of moderate decrease and that its Fourier transform $\hat{f}$ is supported in $I = [-1/2, 1/2]$. Then, f is entirely determined by its restriction to $\mathbb{Z}$. This means that if g is another function of moderate decrease whose Fourier transform is supported in I and $f(n) = g(n)$ for all $n \in \mathbb{Z}$, then $f = g$. More precisely:

(a) Prove that the following reconstruction formula holds:

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) K(x - n) \qquad \text{where } K(y) = \frac{\sin \pi y}{\pi y}.$$

Note that $K(y) = O(1/|y|)$ as $|y| \to \infty$.

(b) If $\lambda > 1$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda} f\left(\frac{n}{\lambda}\right) K_\lambda\left(x - \frac{n}{\lambda}\right) \qquad \text{where } K_\lambda(y) = \frac{\cos \pi y - \cos \pi \lambda y}{\pi^2 (\lambda - 1) y^2}.$$

Thus, if one samples f "more often," the series in the reconstruction formula converges faster since $K_\lambda(y) = O(1/|y|^2)$ as $|y| \to \infty$. Note that $K_\lambda(y) \to K(y)$ as $\lambda \to 1$.

(c) Prove that $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |f(n)|^2.$

[Hint: For part (a) show that if χ is the characteristic function of I, then $\hat{f}(\xi) = \chi(\xi) \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n \xi}$. For (b) use the function in Figure 2 instead of $\chi(\xi)$.]

21. Suppose that f is continuous on $\mathbb{R}$. Show that f and $\hat{f}$ cannot both be compactly supported unless $f = 0$. This can be viewed in the same spirit as the uncertainty principle.

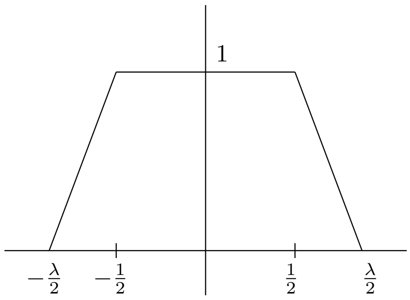

Figure 2. The function in Exercise 20

[Hint: Assume f is supported in $[0, 1/2]$. Expand f in a Fourier series in the interval $[0, 1]$, and note that as a result, f is a trigonometric polynomial.]

22. The heuristic assertion stated before Theorem 4.1 can be made precise as follows. If F is a function on $\mathbb{R}$, then we say that the preponderance of its mass is contained in an interval I (centered at the origin) if

$$\int_I x^2|F(x)|^2\,dx \geq \frac{1}{2}\int_{\mathbb{R}} x^2|F(x)|^2\,dx. \tag{16}$$

Now suppose $f \in \mathcal{S}$, and (16) holds with $F = f$ and $I = I_1$; also with $F = \hat{f}$ and $I = I_2$. Then if L_j denotes the length of I_j, we have

$$L_1 L_2 \geq \frac{1}{2\pi}.$$

A similar conclusion holds if the intervals are not necessarily centered at the origin.

23. The Heisenberg uncertainty principle can be formulated in terms of the operator $L = -\frac{d^2}{dx^2} + x^2$, which acts on Schwartz functions by the formula

$$L(f) = -\frac{d^2 f}{dx^2} + x^2 f.$$

This operator, sometimes called the **Hermite operator**, is the quantum analogue of the harmonic oscillator. Consider the usual inner product on $\mathcal{S}$ given by

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \qquad \text{whenever } f, g \in \mathcal{S}.$$

(a) Prove that the Heisenberg uncertainty principle implies

$$(Lf, f) \geq (f, f) \quad \text{for all } f \in \mathcal{S}.$$

This is usually denoted by $L \geq I$. [Hint: Integrate by parts.]

(b) Consider the operators A and A^* defined on $\mathcal{S}$ by

$$A(f) = \frac{df}{dx} + xf \quad \text{and} \quad A^*(f) = -\frac{df}{dx} + xf.$$

The operators A and A^* are sometimes called the **annihilation** and **creation** operators, respectively. Prove that for all $f, g \in \mathcal{S}$ we have

(i) $(Af, g) = (f, A^*g)$,
(ii) $(Af, Af) = (A^*Af, f) \geq 0$,
(iii) $A^*A = L - I$.

In particular, this again shows that $L \geq I$.

(c) Now for $t \in \mathbb{R}$, let

$$A_t(f) = \frac{df}{dx} + txf \quad \text{and} \quad A_t^*(f) = -\frac{df}{dx} + txf.$$

Use the fact that $(A_t^* A_t f, f) \geq 0$ to give another proof of the Heisenberg uncertainty principle which says that whenever $\int_{-\infty}^{\infty} |f(x)|^2\, dx = 1$ then

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} \left|\frac{df}{dx}\right|^2 dx\right) \geq 1/4.$$

[Hint: Think of $(A_t^* A_t f, f)$ as a quadratic polynomial in t.]

6 Problems

1. The equation

$$x^2 \frac{\partial^2 u}{\partial x^2} + ax\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \tag{17}$$

with $u(x, 0) = f(x)$ for $0 < x < \infty$ and $t > 0$ is a variant of the heat equation which occurs in a number of applications. To solve (17), make the change of variables $x = e^{-y}$ so that $-\infty < y < \infty$. Set $U(y, t) = u(e^{-y}, t)$ and $F(y) = f(e^{-y})$. Then the problem reduces to the equation

$$\frac{\partial^2 U}{\partial y^2} + (1 - a)\frac{\partial U}{\partial y} = \frac{\partial U}{\partial t},$$