(a) If f is T-periodic, continuous, and piecewise C^1 with $\int_0^T f(t)\,dt = 0$, show that

$$\int_0^T |f(t)|^2\,dt \le \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2\,dt,$$

with equality if and only if $f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T)$. [Hint: Apply Parseval's identity.]

(b) If f is as above and g is just C^1 and T-periodic, prove that

$$\left|\int_0^T \overline{f(t)}g(t)\,dt\right|^2 \le \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2\,dt \int_0^T |g'(t)|^2\,dt.$$

(c) For any compact interval $[a,b]$ and any continuously differentiable function f with $f(a) = f(b) = 0$, show that

$$\int_a^b |f(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\,dt.$$

Discuss the case of equality, and prove that the constant $(b-a)^2/\pi^2$ cannot be improved. [Hint: Extend f to be odd with respect to a and periodic of period $T = 2(b-a)$ so that its integral over an interval of length T is 0. Apply part a) to get the inequality, and conclude that equality holds if and only if $f(t) = A\sin(\pi\frac{t-a}{b-a})$].

12. Prove that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

[Hint: Start with the fact that the integral of $D_N(\theta)$ equals 2π, and note that the difference $(1/\sin(\theta/2)) - 2/\theta$ is continuous on $[-\pi,\pi]$. Apply the Riemann-Lebesgue lemma.]

13. Suppose that f is periodic and of class C^k. Show that

$$\hat{f}(n) = o(1/|n|^k),$$

that is, $|n|^k\hat{f}(n)$ goes to 0 as $|n| \to \infty$. This is an improvement over Exercise 10 in Chapter 2.

[Hint: Use the Riemann-Lebesgue lemma.]

14. Prove that the Fourier series of a continuously differentiable function f on the circle is absolutely convergent.

[Hint: Use the Cauchy-Schwarz inequality and Parseval's identity for f'.]

15. Let f be 2π-periodic and Riemann integrable on $[-\pi,\pi]$.

(a) Show that

$$\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-inx}\,dx$$

hence

$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]e^{-inx}\,dx.$$

(b) Now assume that f satisfies a Hölder condition of order α, namely

$$|f(x+h) - f(x)| \leq C|h|^{\alpha}$$

for some $0 < \alpha \leq 1$, some $C > 0$, and all x, h. Use part a) to show that

$$\hat{f}(n) = O(1/|n|^{\alpha}).$$

(c) Prove that the above result cannot be improved by showing that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x},$$

where $0 < \alpha < 1$, satisfies

$$|f(x+h) - f(x)| \leq C|h|^{\alpha},$$

and $\hat{f}(N) = 1/N^{\alpha}$ whenever $N = 2^k$.

[Hint: For (c), break up the sum as follows $f(x+h) - f(x) = \sum_{2^k \leq 1/|h|} + \sum_{2^k > 1/|h|}$. To estimate the first sum use the fact that $|1 - e^{i\theta}| \leq |\theta|$ whenever θ is small. To estimate the second sum, use the obvious inequality $|e^{ix} - e^{iy}| \leq 2$.]

16. Let f be a 2π-periodic function which satisfies a Lipschitz condition with constant K; that is,

$$|f(x) - f(y)| \leq K|x-y| \quad \text{for all } x, y.$$

This is simply the Hölder condition with $\alpha = 1$, so by the previous exercise, we see that $\hat{f}(n) = O(1/|n|)$. Since the harmonic series $\sum 1/n$ diverges, we cannot say anything (yet) about the absolute convergence of the Fourier series of f. The outline below actually proves that the Fourier series of f converges absolutely and uniformly.