Show that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$.

12. Prove that if a series of complex numbers $\sum c_n$ converges to s, then $\sum c_n$ is Cesàro summable to s.

[Hint: Assume $s_n \to 0$ as $n \to \infty$.]

13. The purpose of this exercise is to prove that Abel summability is stronger than the standard or Cesàro methods of summation.

(a) Show that if the series $\sum_{n=1}^{\infty} c_n$ of complex numbers converges to a finite limit s, then the series is Abel summable to s. [Hint: Why is it enough to prove the theorem when $s = 0$? Assuming $s = 0$, show that if $s_N = c_1 + \cdots + c_N$, then $\sum_{n=1}^{N} c_n r^n = (1 - r) \sum_{n=1}^{N} s_n r^n + s_N r^{N+1}$. Let $N \to \infty$ to show that

$$\sum c_n r^n = (1 - r) \sum s_n r^n.$$

Finally, prove that the right-hand side converges to 0 as $r \to 1$.]

(b) However, show that there exist series which are Abel summable, but that do not converge. [Hint: Try $c_n = (-1)^n$. What is the Abel limit of $\sum c_n$?]

(c) Argue similarly to prove that if a series $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to σ, then it is Abel summable to σ. [Hint: Note that

$$\sum_{n=1}^{\infty} c_n r^n = (1 - r)^2 \sum_{n=1}^{\infty} n \sigma_n r^n,$$

and assume $\sigma = 0$.]

(d) Give an example of a series that is Abel summable but not Cesàro summable. [Hint: Try $c_n = (-1)^{n-1} n$. Note that if $\sum c_n$ is Cesàro summable, then c_n / n tends to 0.]

The results above can be summarized by the following implications about series:

$$\text{convergent} \Longrightarrow \text{Cesàro summable} \Longrightarrow \text{Abel summable},$$

and the fact that none of the arrows can be reversed.

14. This exercise deals with a theorem of Tauber which says that under an additional condition on the coefficients c_n, the above arrows can be reversed.

(a) If $\sum c_n$ is Cesàro summable to σ and $c_n = o(1/n)$ (that is, $nc_n \to 0$), then $\sum c_n$ converges to σ. [Hint: $s_n - \sigma_n = [(n-1)c_n + \cdots + c_2]/n$.]

(b) The above statement holds if we replace Cesàro summable by Abel summable. [Hint: Estimate the difference between $\sum_{n=1}^{N} c_n$ and $\sum_{n=1}^{N} c_n r^n$ where $r = 1 - 1/N$.]

15. Prove that the Fejér kernel is given by

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

[Hint: Remember that $NF_N(x) = D_0(x) + \cdots + D_{N-1}(x)$ where $D_n(x)$ is the Dirichlet kernel. Therefore, if $\omega = e^{ix}$ we have

$$NF_N(x) = \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1-\omega}.]$$

16. The Weierstrass approximation theorem states: Let f be a continuous function on the closed and bounded interval $[a, b] \subset \mathbb{R}$. Then, for any $\epsilon > 0$, there exists a polynomial P such that

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon.$$

Prove this by applying Corollary 5.4 of Fejér's theorem and using the fact that the exponential function e^{ix} can be approximated by polynomials uniformly on any interval.

17. In Section 5.4 we proved that the Abel means of f converge to f at all points of continuity, that is,

$$\lim_{r\to 1} A_r(f)(\theta) = \lim_{r\to 1}(P_r * f)(\theta) = f(\theta), \qquad \text{with } 0 < r < 1,$$

whenever f is continuous at θ. In this exercise, we will study the behavior of $A_r(f)(\theta)$ at certain points of discontinuity.

An integrable function is said to have a **jump discontinuity** at θ if the two limits

$$\lim_{\substack{h \to 0 \\ h > 0}} f(\theta + h) = f(\theta^+) \qquad \text{and} \qquad \lim_{\substack{h \to 0 \\ h > 0}} f(\theta - h) = f(\theta^-)$$

exist.

(a) Prove that if f has a jump discontinuity at θ, then

$$\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}, \qquad \text{with } 0 \le r < 1.$$

[Hint: Explain why $\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{\pi} P_r(\theta)\,d\theta = \frac{1}{2}$, then modify the proof given in the text.]

(b) Using a similar argument, show that if f has a jump discontinuity at θ, the Fourier series of f at θ is Cesàro summable to $\frac{f(\theta^+)+f(\theta^-)}{2}$.

18. If $P_r(\theta)$ denotes the Poisson kernel, show that the function

$$u(r,\theta) = \frac{\partial P_r}{\partial \theta},$$

defined for $0 \le r < 1$ and $\theta \in \mathbb{R}$, satisfies:

(i) $\triangle u = 0$ in the disc.

(ii) $\lim_{r\to 1} u(r,\theta) = 0$ for each θ.

However, u is not identically zero.

19. Solve Laplace's equation $\triangle u = 0$ in the semi infinite strip

$$S = \{(x,y) : 0 < x < 1,\ 0 < y\},$$

subject to the following boundary conditions

$$\begin{cases} u(0,y) = 0 & \text{when } 0 \le y, \\ u(1,y) = 0 & \text{when } 0 \le y, \\ u(x,0) = f(x) & \text{when } 0 \le x \le 1 \end{cases}$$

where f is a given function, with of course $f(0) = f(1) = 0$. Write

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and expand the general solution in terms of the special solutions given by

$$u_n(x,y) = e^{-n\pi y}\sin(n\pi x).$$

Express u as an integral involving f, analogous to the Poisson integral formula (6).

20. Consider the Dirichlet problem in the annulus defined by $\{(r,\theta) : \rho < r < 1\}$, where $0 < \rho < 1$ is the inner radius. The problem is to solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to the boundary conditions

$$\begin{cases} u(1,\theta) = f(\theta), \\ u(\rho,\theta) = g(\theta), \end{cases}$$