Assignment 9

Problem 1. We have seen that the kernel of a bounded operator is closed. For unbounded operators, this is not always the case.

(a) Give an example of an operator whose kernel is not closed.

What is the relation between closed operators, and operators with closed kernels?

(b) Show that the kernel of a closed operator is closed.

(c) Give an example of an operator with closed kernel which is not closed.

Problem 2. Let X, Y be Hilbert spaces, and $U : X \to Y$ a unitary map. Let $T : D(T) \to X$ be a densely-defined operator in X. We define the operator $\tilde{T}$ in Y by

$$\begin{aligned} &D(\tilde{T}) = UD(T) = \{Ux : x \in D(T)\}; \\ &\tilde{T} = UTU^{-1}. \end{aligned}$$

The goal of this exercise is to observe that T and $\tilde{T}$ are closely related. Precisely, show that

(a) $D(\tilde{T})$ is dense in Y, and $D(\tilde{T}) = Y$ iff $D(T) = X$.

(b) $\|\tilde{T}\| = \|T\|$ (both may be infinite).

(c) $\tilde{T}$ is closed iff T is closed. Also, $U\overline{T}U^{-1} = \overline{\tilde{T}}$.

(d) $D(\tilde{T}^*) = UD(T^*)$ and $UT^*U^{-1} = \tilde{T}^*$.

(e) $\tilde{T}$ is symmetric iff T is symmetric, and $\tilde{T}$ is self-adjoint iff T is self-adjoint.

(f) $\rho(\tilde{T}) = \rho(T)$; $\sigma_{\rm p}(\tilde{T}) = \sigma_{\rm p}(T)$; $\sigma_{\rm c}(\tilde{T}) = \sigma_{\rm c}(T)$; $\sigma_{\rm r}(\tilde{T}) = \sigma_{\rm r}(T)$.

Problem 3. We have seen in Assignment 5 that the Fourier functions $e_k(x) = \frac{1}{\sqrt{2\pi}} {\rm e}^{{\rm i}kx}$ form an orthonormal basis for $L^2(\mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus $[0, 2\pi]$. Then any $f \in L^2(\mathbb{T})$ can be written as

$$f = \sum_{k \in \mathbb{Z}} \widehat{f}_k e_k.$$

The Fourier coefficients $\widehat{f}_k$ are uniquely determined (actually, $\widehat{f}_k = (e_k, f)$) and they satisfy $\sum_k |\widehat{f}_k|^2 = \|f\|^2 < \infty$. Thus the Fourier transform can be viewed as a map

$$\begin{aligned} U : & L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}), \\ & f \mapsto Uf = (\widehat{f}_k). \end{aligned}$$

(a) Check that U is a unitary map.

(b) If $f \in C^1(\mathbb{T})$, check that

$$(\widehat{f'})_k = \mathrm{i}k\widehat{f}_k.$$

Let $D = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ the differential operator with domain $D(D) = C^1(\mathbb{T})$, and let M be the multiplication operator in $\ell^2(\mathbb{Z})$, $M(a_k) = (ka_k)$, with domain $UD(D)$. Show that

(c) $M = U^{-1}DU$.

(d) D and M are symmetric.

(e) Describe the closure $\overline{M}$, and check that $\overline{M}$ is self-adjoint.

(f) Conclude that D and M are both essentially self-adjoint.