

Assignment 8

Problem 1. Let $T : D(T) \rightarrow X$ be densely-defined, and let $\sigma'(T)$ denote the set of *approximate eigenvalues*; precisely,

$$\sigma'(T) = \left\{ \lambda \in \mathbb{C} : \inf_{\substack{x \in D(T) \\ \|x\|=1}} \|(T - \lambda \mathbb{1})x\| = 0 \right\}.$$

- (a) Show that, if T is self-adjoint, $\sigma'(T) = \sigma(T)$. (Compare with Prop. 5.11 for bounded operators.)
- (b) More generally, show that

$$\sigma_p(T) \cup \sigma_c(T) \subset \sigma'(T) \subset \sigma(T).$$

Problem 2. An example where the domain affects the spectrum. Let $X = \ell^2(\mathbb{N})$, $x = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, and $M = \text{span}\{x\}$. Let P be the orthogonal projection onto M .

- (a) Suppose that $D(P) = \ell^2$. Show that $1 \in \sigma_p(P)$.
- (b) Suppose that $D(P)$ is the space of elements of ℓ^2 with finitely many non zero entries. Show that $1 \in \sigma_r(P)$.

Problem 3. Generalisation of the above example. Let $S \subset T$ be densely-defined operators, with T an extension of S . Prove that

$$\begin{aligned} \sigma_p(S) &\subset \sigma_p(T); \\ \sigma_r(S) &\supset \sigma_r(T); \\ \sigma_c(S) &\subset \sigma_p(T) \cup \sigma_c(T). \end{aligned}$$

Problem 4. This question is optional, but it is puzzling.

Let ℓ_0 be the space of all sequences of complex numbers (x_1, x_2, \dots) with finitely many nonzero entries. Can you find a norm such that ℓ_0 is complete? If yes, give it. If not, prove there exists none.

(I heard that Baire category theorem might help.)