Assignment 6

Problem 1. Let X be the Banach space $C([0,1])$ with the sup norm, and $T : X \to X$ be the operator

$$(Tf)(x) = f(x) + \int_0^x f(y)\mathrm{d}y.$$

Show that $\operatorname{ran} T = X$ and $\ker T = \{0\}$.

Hint: Replace the integral equation by a differential equation, and use appropriate theorems about existence and unicity of solutions.

Problem 2. (Projections)

(a) Let M be a closed subspace of a Hilbert space X, and let P be the orthogonal projector onto M. Show that $P^2 = P$, $\|P\| = 1$ (if $M \neq \{0\}$), and that

$$(Px, y) = (x, Py)$$

for any $x, y \in X$.

(b) Conversely, suppose that $P : X \to X$ is linear, and satisfies $P^2 = P$ and $(Px, y) = (x, Py)$ for all $x, y \in X$. Show that P is the orthogonal projection onto some closed subspace.

Problem 3. Let T be the multiplication operator by a function g. That is, we define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$Tf(x) = g(x)f(x),$$

where g is a fixed function, that we suppose to be continuous and bounded. Prove that

(a) T is bounded;

(b) the spectrum is $\sigma(T) = \overline{\{g(x) : x \in \mathbb{R}\}}$;

(c) give an example of a continuous, bounded function g such that T has eigenvalues;

(d) can you find a function $g \not\equiv 0$ such that T is compact?

Problem 4. (Unitary operators)

(a) Let $U : X_1 \to X_2$ be a unitary operator between Hilbert spaces. Show that $\|U\| = 1$ and $U^* = U^{-1}$.

(b) Let $\mathcal{S}_{\mathbb{N}}$ be the group of permutations (bijections) $\mathbb{N} \to \mathbb{N}$. For $\pi \in \mathcal{S}_{\mathbb{N}}$, define $U_\pi : \ell^2(\mathbb{N}, \mathbb{C}) \to \ell^2(\mathbb{N}, \mathbb{C})$ by

$$U_\pi(x_1, x_2, \dots) = (x_{\pi(1)}, x_{\pi(2)}, \dots).$$

Show that U_π is unitary. Show that this *unitary representation* of $\mathcal{S}_{\mathbb{N}}$ preserves the group structure, in the sense that

$$U_\pi U_{\pi'} = U_{\pi \circ \pi'}.$$

Problem 5. Let T be a bounded operator on a Hilbert space X.

(a) Suppose that (x, Tx) is real for all $x \in X$. Show that T is symmetric.

We suppose in addition that T is positive, $(x, Tx) > 0$ for all $x \in X$.

(b) Show that (x, Ty), $x, y \in X$, is an inner product.

(c) Suppose there exists $c > 0$ such that $(x, Tx) \geq c\|x\|^2$ for all $x \in X$. Show that X equipped with the new inner product is a Hilbert space.

(d) Suppose there exists no $c > 0$ such that $(x, Tx) \geq c\|x\|^2$ for all $x \in X$. Show that X equipped with the new inner product is not complete.

This question is not easy, a hint may be necessary. Consider the inclusion map

$$\iota \ : \quad \big(X, (\cdot, \cdot)\big) \to \big(X, (\cdot, T\cdot)\big)$$
$$x \mapsto x,$$

and use the inverse mapping theorem.