Assignment 4

Problem 1. Recall that a map between topological spaces is open iff the image of each open set is open. Show that a linear map between normed spaces is open iff the image of the unit ball (around 0) contains a ball around 0.

Problem 2. Let X, Y, Z be normed spaces, and $S : X \to Y$ and $T : Y \to Z$ be operators.

(a) Show that $\|T \circ S\| \leq \|S\| \, \|T\|$.

(b) Give an example where $\|T \circ S\| < \|S\| \, \|T\|$.

Problem 3. Let ℓ_{f} be the space of sequences $x = (x_1, x_2, \dots)$ of complex numbers with finitely many nonzero terms. We consider the norm $\|x\| = \sup_i |x_i|$. Define $T : \ell_{\mathrm{f}} \to \ell_{\mathrm{f}}$ by

$$Tx = (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots)$$

Show that T is linear and bounded. Show that T is bijective but that T^{-1} is unbounded. Does this contradict the Inverse Mapping Theorem?

Problem 4. Let $T : X \to Y$ be a bounded operator between Banach spaces X and Y. Show that, if T is bijective, there exist constants c_1 and c_2 such that

$$c_1\|x\| \leq \|Tx\| \leq c_2\|x\|$$

for all $x \in X$.

Problem 5. Let ℓ_{f} be the space of sequences of complex numbers with finitely many nonzero entries, equipped with the ℓ^2 norm.

(a) Is ℓ_{f} complete? Prove your answer.

Let $M : \ell_{\mathrm{f}} \to \ell^2$ the multiplication operator such that

$$M(a_1, a_2, a_3, \dots) = (a_1, 2a_2, 3a_3, \dots).$$

(b) Is M bounded or unbounded? Prove your answer.

(c) Is M closed or not closed? Prove your answer.