Assignment 3

Problem 1.

(a) Use the Hahn-Banach theorem to show that, if x_n converges weakly to x, then
$$\liminf_{n\to\infty} \|x_n\| \geq \|x\|.$$
Hint: Consider a functional f such that $\|f\| = 1$ and $f(x) = \|x\|$.

(b) Find examples in ℓ^p and L^p of weakly convergent sequences, where the inequality above is strict.

Problem 2. Let X be a normed linear space. Show the following:

(a) If $\|x_n - x\| \to 0$, then $x_n \rightharpoonup x$ weakly.

(b) If X is finite-dimensional, then weak convergence is equivalent to norm convergence.

(c) If X is infinite-dimensional, then weak convergence *does not* imply norm convergence. Give an example.

(d) If x_n converges weakly to both x and y, then $x = y$.

Problem 3. Show that $(\ell^p)^* = \ell^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, when $1 \leq p < \infty$. More precisely, show that

(a) any $a \in \ell^q$ defines a continuous linear functional f_a on ℓ^p by $f_a(x) = \sum_n a_n x_n$, $x \in \ell^p$;

(b) to any functional $f \in (\ell^p)^*$ there corresponds a sequence $a \in \ell^q$ such that $f = f_a$;

(c) the operator norm of f_a is equal to the ℓ^q norm of a.

Are the ℓ^p spaces reflexive?

Problem 4. Show that the dual of $L^\infty([0,1])$ is larger than $L^1([0,1])$. For this, you may consider the following steps:

(a) Consider the subspace $C([0,1]) \subset L^\infty([0,1])$, and introduce the map $\varphi : C([0,1]) \to \mathbb{C}$ such that

$$\varphi(f) = f(0).$$

Show that there exists a bounded linear functional $\overline{\varphi}$ on L^∞ that is an extension of φ.

(b) Consider the functions $f_n(s) = \mathrm{e}^{-ns}$ in L^∞. Show that there are no functions $g \in L^1([0,1])$ such that

$$\overline{\varphi}(f_n) = \int_0^1 f_n(x) g(x) \mathrm{d}x$$

for all n.

(c) Conclude that there exist no functions $g \in L^1([0,1])$ that represents $\overline{\varphi}$.

Problem 5. Find an example of $1 \leq p, q < \infty$ and a sequence (x_n) such that (x_n) converges strongly to 0 in ℓ^p, and weakly (but not strongly) in ℓ^q.