

Assignment 2

Problem 1. (Quotient space) Let X be a normed vector space and Z be a subspace of X . It induces an equivalence relation, $x \sim y$ iff $x - y \in Z$.

- (a) Check that the set of equivalence classes of X , X/Z , is a linear space when equipped with the following linear operations:

$$\alpha[x] + \beta[y] = [\alpha x + \beta y].$$

- (b) Suppose that Z is closed, and define the **quotient norm** on X/Z by

$$\|[x]\|_q = \inf\{\|y\| : y \sim x\} = \inf\{\|x + z\| : z \in Z\}.$$

Check that $\|\cdot\|_q$ is a norm. Why do we need Z to be closed?

Problem 2.

- (a) Check that the operator norm is a norm indeed.
 (b) Prove Theorem 1.5 (a linear map is continuous iff it is bounded).
 (c) Prove that $\|Ax\| \leq \|A\|\|x\|$.
 (d) Prove that the norm is continuous, i.e., $\|x_n\| \rightarrow \|x\|$ if $x_n \rightarrow x$.

Problem 3. In the linear space $C^\infty([0, 1])$ with the sup norm, we consider the two operators M (multiplication) and D (differentiation):

$$(Mf)(x) = xf(x), \quad (Df)(x) = f'(x).$$

Show that $\|M\| = 1$ and $\|D\| = \infty$.

Problem 4. Prove that the integral operator K on $C([0, 1])$ defined by

$$(Kf)(x) = \int_0^1 k(x, y)f(y)dy,$$

where $k \in C([0, 1] \times [0, 1])$, has norm

$$\|K\| = \max_{0 \leq x \leq 1} \int_0^1 |k(x, y)|dy.$$

Problem 5. Prove the claims of Proposition 1.8 about finite-dimensional Banach spaces.

Problem 6. Consider the functional $\delta : C([0, 1]) \rightarrow \mathbb{R}$ such that $\delta(f) = f(0)$. Find the norm of δ , when $C([0, 1])$ is equipped with the L^p norm, $1 \leq p \leq \infty$.