

Assignment 1

Problem 1. Prove that the Banach space $C([0, 1], \mathbb{R})$, with the $\|\cdot\|_\infty$ norm, is separable.

Problem 2. Let $Y \subset \ell^p(\mathbb{C})$ be the set of sequences (of complex numbers) with finitely many non-zero entries, i.e.

$$Y = \{(x_1, x_2, \dots) : \exists k < \infty \text{ such that } x_i = 0 \forall i > k\}$$

We equip Y with the usual ℓ^p norm, $1 \leq p \leq \infty$. Explain why Y is not a Banach space. (Prove your assertions!) Show that Y is dense in $\ell^p(\mathbb{C})$.

Problem 3.

- (a) Show that $\ell^\infty(\mathbb{C})$ is *not* separable. (*Hint:* This is similar to proving that it is not countable.)
- (b) Let $Z \subset \ell^\infty(\mathbb{C})$ be the set of sequences $x = (x_1, x_2, \dots)$ such that $\lim_{k \rightarrow \infty} x_k = 0$. Show that Z is a closed subspace.

Problem 4.

- (a) Show that $\ell^p(\mathbb{C}) \subsetneq \ell^q(\mathbb{C})$ when $1 \leq p < q \leq \infty$.
- (b) Find a sequence of numbers converging to 0, which is *not* in any ℓ^p space with $1 \leq p < \infty$.
- (c) If $x \in \ell^p$ for some finite p , show that

$$\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty.$$

Problem 5. Consider two arbitrary norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in \mathbb{R}^n . Show that a sequence (x_k) converges with respect to $\|\cdot\|_1$ iff it converges with respect to $\|\cdot\|_2$. Are the limits necessarily the same?

Hint: Show that, if $\|x_k\| \rightarrow 0$, then each component of x_k goes to 0. This holds with respect to any norm.