Assignment 6 — Solutions

Problem 2.
(a) We have

$$\|Tf\|^2 = \int |g(x)|^2 |f(x)|^2 \mathrm{d}x \le \sup |g(x)|^2 \|f\|^2,$$

so that $\|T\| \le \sup |g(x)|$. It will actually follow from (b) that $\|T\| = \sup |g(x)|$.
(b) Let us first see that if $\lambda \notin \overline{\{g(x) : x \in X\}}$, we have $\lambda \in \rho(T)$. Consider the operator S defined by

$$Sf(x) = \frac{1}{g(x) - \lambda} f(x).$$

It is bounded since $\|S\| \le \sup \frac{1}{|g(x)-\lambda|} < \infty$. One also sees that

$$S(T - \lambda) = (T - \lambda)S = \mathbb{1}.$$

Then $T - \lambda$ has a bounded inverse, so that $\lambda \in \rho(T)$.

We prove now that if $\lambda \in \{g(x) : x \in X\}$, we have $\lambda \in \sigma(T)$. Since the spectrum of a bounded operator is closed (Corollary 5.8), and together with the result that we just proved, we obtain that $\overline{\{g(x) : x \in X\}} = \sigma(T)$.

We use the following property. For any bounded operator,

$$\inf_{\|x\|=1} \|(T - \lambda)x\| = 0 \quad \Longrightarrow \quad \lambda \in \sigma(T).$$

We actually proved in the course (Propostion 5.12) that the two properties above are *equivalent* for *self-adjoint* operators. But the implication holds for general bounded operators.

Let x_0 such that $g(x_0) = \lambda$, and consider the functions f_n defined by

$$f_n(x) = \begin{cases} \sqrt{n} & \text{if } |x - x_0| < \frac{1}{2n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f_n\| = 1$, and

$$\|(T - \lambda)f_n\|^2 = n \int_{x_0 - \frac{1}{2n}}^{x_0 + \frac{1}{2n}} |g(x) - \lambda|^2 \mathrm{d}x \le \sup_{|x - x_0| < \frac{1}{2n}} |g(x) - \lambda|^2.$$

The latter goes to 0 as $n \to \infty$ since g is continuous. Then $\lambda \in \sigma(T)$.
(c) A trivial example: $g(x) \equiv 1$, so that T is the identity and 1 is the (unique) eigenvalue. More generally, λ is an eigenvalue iff $g^{-1}(\{\lambda\})$ has nonzero Lebesgue measure, in which case any function whose support is in $g^{-1}(\{\lambda\})$ is an eigenvector.
(d) T is not compact, unless $g \equiv 0$. Suppose that $|g(x)| > \varepsilon$ for all x in a neighbourhood of x_0, and consider the functions f_n defined above. We show that (Tf_n)

has no converging subsequence. For $m < n$ large enough, we have

$$\begin{aligned}\|Tf_m - Tf_n\|^2 &= \int_{|x|<\frac{1}{2n}} |g(x)|^2 (n-m)\mathrm{d}x + \int_{\frac{1}{2n}<|x|<\frac{1}{2m}} |g(x)|^2 m\mathrm{d}x \\ &> \varepsilon^2 \left[\tfrac{1}{n}(n-m) + (\tfrac{1}{m} - \tfrac{1}{n})m\right] \\ &= (1 - \tfrac{m}{n}) 2\varepsilon^2.\end{aligned}$$

Suppose that (Tf_{n_k}) is a converging subsequence. Then it is Cauchy and $\|Tf_{n_k} - Tf_{n_{k'}}\|$ is as small as we want for k, k' large enough. But the above bound implies that

$$\|Tf_{n_k} - Tf_{n_{k'}}\| > (1 - \tfrac{n_k}{n_{k'}}) 2\varepsilon^2,$$

which is not small if $n_{k'} > 2n_k$. Then (Tf_{n_k}) does not converge.