Assignment 8

Problem 1. Prove Theorem 6.7. That is, if T is a densely defined operator, prove that

(a) T^* is closed.
(b) If T^* is densely defined, then T is closable.
(c) If T is symmetric, then T is closable and $(\overline{T})^* = T^*$.
(d) If $T : D(T) \to X$ is bijective and closed, then T^{-1} is bounded. (Hint: show that T^{-1} is closed and use the closed graph theorem).

Problem 2. Let $T : D(T) \to X$ be densely-defined, and let $\sigma'(T)$ denote the set of *approximate eigenvalues*; precisely,

$$\sigma'(T) = \Big\{\lambda \in \mathbb{C} : \inf_{\substack{x \in D(T) \\ \|x\|=1}} \|(T - \lambda \mathbb{1})x\| = 0\Big\}.$$

(a) Show that, if T is self-adjoint, $\sigma'(T) = \sigma(T)$. (Compare with Prop. 5.12 for bounded operators.)
(b) More generally, show that

$$\sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T) \subset \sigma'(T) \subset \sigma(T).$$

Problem 3. An example where the domain affects the spectrum. Let $X = \ell^2(\mathbb{N})$, $x = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, and $M = \mathrm{span}\{x\}$. Let P be the orthogonal projection onto M.

(a) Suppose that $D(P) = \ell^2$. Show that $1 \in \sigma_{\mathrm{p}}(P)$.
(b) Suppose that $D(P)$ is the space of elements of ℓ^2 with finitely many non zero entries. Show that $1 \in \sigma_{\mathrm{r}}(P)$.

Problem 4. Generalisation of the above example. Let $S \subset T$ be densely-defined operators, with T an extension of S. Prove that

$$\begin{aligned} &\sigma_{\mathrm{p}}(S) \subset \sigma_{\mathrm{p}}(T); \\ &\sigma_{\mathrm{r}}(S) \supset \sigma_{\mathrm{r}}(T); \\ &\sigma_{\mathrm{c}}(S) \subset \sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{c}}(T). \end{aligned}$$

Problem 5. This question is optional, but it is puzzling.

Let ℓ_0 be the space of all sequences of complex numbers $(x_1, x_2, \dots)$ with finitely many nonzero entries. Can you find a norm such that ℓ_0 is complete? If yes, give it. If not, prove there exists none.

(I heard that Baire category theorem might help.)