Assignment 7

Problem 1. (Shift operators) We consider the right and left shift operators on $\ell^2(\mathbb{N})$:

$$S(x_1, x_2, \dots) = (0, x_1, x_2, \dots),$$
$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

(a) Find $\|S\|$, $\|T\|$, S^*, T^*, S^{-1}, T^{-1}.
(b) Find $\operatorname{ran} S$, $\operatorname{ran} T$, $\ker S$, $\ker T$, and check that

$$\operatorname{ran} S = (\ker T)^\perp, \qquad \operatorname{ran} T = (\ker S)^\perp.$$

(c) Find the spectrum of S and T.

Problem 2. Let $T \in \mathcal{B}(X)$, and $\alpha, \beta \in \rho(T)$. Let $R_\alpha = (T - \alpha \mathbb{1})^{-1}$ denote the resolvent.

(a) Show that it satisfies the *Hilbert relation* (or *resolvent equation*)

$$R_\alpha - R_\beta = (\alpha - \beta) R_\alpha R_\beta.$$

(b) Show that $R_\alpha R_\beta = R_\beta R_\alpha$.

Problem 3. Let X be a separable Hilbert space. A bounded operator $T : X \to X$ is *Hilbert-Schmidt* if there exists an orthonormal basis (e_n) such that $\sum_n \|Te_n\|^2 < \infty$.

(a) Show that Hilbert-Schmidt operators are compact.

We define the norm of a Hilbert-Schmidt operator T by

$$\|T\|_{\mathrm{HS}} = \Big(\sum_{n \geq 1} \|Te_n\|^2 \Big)^{1/2}.$$

(b) Show that $\| \cdot \|_{\mathrm{HS}}$ is a norm.
(c) Show that the Hilbert-Schmidt norm does not depend on the choice of the orthonormal basis.

Let us now remove the assumption that T is bounded. We still suppose that $\sum_n \|Te_n\|^2 < \infty$ for some orthonormal basis (e_n).

(d) Show that T is not necessarily bounded. (*Hint:* Construct a suitable operator using a Hamel basis that contains (e_n). Thanks to Michael Doré for the hint!)

Problem 4. Consider the integral operator $K : L^2([0,1]) \to L^2([0,1])$ with integral kernel $k(t,s)$, i.e.

$$Kf(t) = \int_0^1 k(t,s) f(s) \mathrm{d}s.$$

Show that its Hilbert-Schmidt norm is

$$\|K\|_{\mathrm{HS}} = \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}s \, |k(t,s)|^2.$$

We now study a compact operator that is not self-adjoint, and whose spectrum consists of $\{0\}$ only. This suggests that the spectral decomposition for self-adjoint compact operators (Thm 5.16) does not have a simple extension to non self-adjoint operators.

Problem 5. Let $K : L^2([0,1]) \to L^2([0,1)]$ be the integral operator defined by

$$Kf(t) = \int_0^t f(s)\,\mathrm{d}s.$$

(a) Find the adjoint operator K^*.
(b) Use Problems 3 and 4 to show that K is Hilbert-Schmidt with $\|K\|_{\mathrm{HS}} = \frac{1}{\sqrt{2}}$. Then K is compact.
(c) Show that $\|K\| = \frac{2}{\pi}$. (Hint: Find the eigenvectors and eigenvalues of the bounded self-adjoint operator K^*K.)
(d) Show that $0 \in \sigma_{\mathrm{c}}(K)$.
(e) Show that $\sigma(K) = \sigma_{\mathrm{c}}(K) = \{0\}$.