Assignment 5

Problem 1.

(a) Show that the norm induced by an inner product satisfies the parallelogram identity.

(b) Let $\|\cdot\|$ be a norm that satisfies the parallelogram identity. Show that the polarization identity defines an inner product. (Hint: You may want to establish the following identity:

$$\|x+y+z\|^2 - \|x-y-z\|^2 = \|x+y\|^2 - \|x-y\|^2 + \|x+z\|^2 - \|x-z\|^2.$$

This may help to prove linearity. As far as I know, this exercise is not easy.)

Problem 2. Show that $L^p(\mathbb{R})$ can be turned into a Hilbert space if and only if $p = 2$, in which case the inner product is

$$(f, g) = \int_{-\infty}^{\infty} \overline{f(x)} g(x) \mathrm{d}x.$$

(Hint: Consider two functions with disjoint supports. Then the parallelogram identity reduces to showing that $(a^p + b^p)^{2/p} = a^2 + b^2$ for any positive numbers a, b.)

Problem 3. Let X be a separable infinite-dimensional Hilbert space. Show that:

(a) Every orthonormal sequence converges weakly to 0.

(b) The unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \leq 1\}$.

(Note: These properties also hold for nonseparable Hilbert spaces.)

In Problems 4 and 5 we prove that the Fourier functions form an orthonormal basis of $L^2([0, 2\pi])$. Fourier series are then a special case of Proposition 4.7. Problem 6 is a simple, cute, nontrivial, and rather surprising application of Fourier series to the Riemann zeta function.

Problem 4. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denote the circle, where operations $x + y$ and $x - y$ are taken modulo 2π. Recall that the convolution of two functions is defined by

$$(f * g)(x) = \int_{\mathbb{T}} f(x - y) g(y) \mathrm{d}y.$$

Let $\varphi_n \geq 0$ be a function on $\mathbb{T}$ that satisfies

$$\int_{\mathbb{T}} \varphi_n(x)\mathrm{d}x = 1, \qquad \text{and} \qquad \lim_{n\to\infty} \int_{|x|>\delta} \varphi_n(x)\mathrm{d}x = 0$$

for any $\delta > 0$. Show that, for any $f \in C(\mathbb{T})$, $\varphi_n * f$ converges uniformly to f, i.e. that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{T}} \big|(\varphi_n * f)(x) - f(x)\big| = 0.$$

Problem 5.

(a) Show that the function $\varphi_n(x) = c_n(1+\cos x)^n$, where c_n is chosen so that $\int_{\mathbb{T}} \varphi_n = 1$, satisfies the properties of Problem 4.

(b) Check that $\{\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}kx}\}_{k\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{T})$.

(c) Show that $\{\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}kx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$. For this, show that

$$\varphi_n(x) = \sum_{k=-n}^{n} a_{nk}\,\mathrm{e}^{\mathrm{i}kx}, \qquad \text{with} \qquad a_{nk} = 2^{-n} c_n \tbinom{2n}{n+k}.$$

Then show that for any $f \in C(\mathbb{T})$,

$$(\varphi_n * f)(x) = \sum_{k=-n}^{n} b_{nk}\,\mathrm{e}^{\mathrm{i}kx}, \qquad \text{with} \qquad b_{nk} = a_{nk} \int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}kx} f(y)\mathrm{d}y.$$

Then use Problem 4 and Proposition 4.7 of the course to conclude that $\{\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}kx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis.

Problem 6. Show that $\zeta(2) \equiv \sum_{n\geq 1} n^{-2} = \pi^2/6$. Hint: Use the fact that $\{\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}kx}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, and that the function $f(x) = x$ has Fourier coefficients proportional to $1/n$.

Can you get another identity with $f(x) = x^2$?