Assignment 2

Problem 1. (Quotient space) Let X be a normed vector space and Z be a subspace of X. It induces an equivalence relation, $x \sim y$ iff $x - y \in Z$.

(a) Check that the set of equivalence classes of X, X/Z, is a linear space when equipped with the following linear operations:

$$\alpha[x] + \beta[y] = [\alpha x + \beta y].$$

(b) Suppose that Z is closed, and define the **quotient norm** on X/Z by

$$\big\|[x]\big\|_{\mathrm{q}} = \inf\big\{\|y\| : y \sim x\big\} = \inf\big\{\|x + z\| : z \in Z\big\}.$$

Check that $\|\cdot\|_{\mathrm{q}}$ is a norm. Why do we need Z to be closed?

Problem 2.

(a) Check that the operator norm is a norm indeed.
(b) Prove Theorem 1.4 (a linear map is continuous iff it is bounded).
(c) Prove that $\|Ax\| \le \|A\|\|x\|$, and that $\|x_n\| \to \|x\|$ if $x_n \to x$.

Problem 3. In the linear space $C^\infty([0,1])$ with the sup norm, we consider the two operators M (multiplication) and D (differentiation):

$$(Mf)(x) = xf(x), \qquad (Df)(x) = f'(x).$$

Show that $\|M\| = 1$ and $\|D\| = \infty$.

Problem 4. Prove that the integral operator K on $C([0,1])$ defined by

$$(Kf)(x) = \int_0^1 k(x,y)f(y)\mathrm{d}y,$$

where $k \in C([0,1] \times [0,1])$, has norm

$$\|K\| = \max_{0\le x\le 1} \int_0^1 |k(x,y)|\mathrm{d}y.$$

Problem 5. Prove the claims of Proposition 1.7 about finite-dimensional Banach spaces.

Problem 6. Consider the functional $\delta : C([0,1]) \to \mathbb{R}$ such that $\delta(f) = f(0)$. Find the norm of δ, when $C([0,1])$ is equipped with the L^p norm, $1 \le p \le \infty$.