Assignment 9

Problem 1. Let X, Y be Hilbert spaces, and $U : X \to Y$ a unitary map. Let $T : D(T) \to X$ be a densely-defined operator in X. We define the operator $\tilde{T}$ in Y by

$$D(\tilde{T}) = UD(T) = \{Ux : x \in D(T)\};$$
$$\tilde{T} = U^{-1}TU.$$

The goal of this exercise is to observe that T and $\tilde{T}$ are closely related. Precisely, show that

(a) $D(\tilde{T})$ is dense in Y, and $D(\tilde{T}) = Y$ iff $D(T) = X$.
(b) $\|\tilde{T}\| = \|T\|$ (both may be infinite).
(c) $\tilde{T}$ is closed iff T is closed. Also, $U^{-1}\overline{T}U = \overline{\tilde{T}}$.
(d) $D(\tilde{T}^*) = UD(T^*)$ and $U^{-1}T^*U = \tilde{T}^*$.
(d) $\tilde{T}$ is symmetric iff T is symmetric, and T is self-adjoint iff T is self-adjoint.
(e) $\rho(\tilde{T}) = \rho(T)$; $\sigma_{\mathrm{p}}(\tilde{T}) = \sigma_{\mathrm{p}}(T)$; $\sigma_{\mathrm{c}}(\tilde{T}) = \sigma_{\mathrm{c}}(T)$; $\sigma_{\mathrm{r}}(\tilde{T}) = \sigma_{\mathrm{r}}(T)$.

Problem 2. We have seen in Assignment 5 that the Fourier functions $e_k(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}kx}$ form an orthonormal basis for $L^2(\mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus $[0, 2\pi]$. Then any $f \in L^2(\mathbb{T})$ can be written as

$$f = \sum_{k \in \mathbb{Z}} \widehat{f}_k e_k.$$

The Fourier coefficients $\widehat{f}_k$ are uniquely determined (actually, $\widehat{f}_k = (e_k, f)$) and they satisfy $\sum_k |\widehat{f}_k|^2 = \|f\|^2 < \infty$. Thus the Fourier transform can be viewed as a map

$$U : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}),$$
$$f \mapsto Uf = (\widehat{f}_k).$$

(a) Check that U is a unitary map.
(b) If $f \in C^1(\mathbb{T})$, check that

$$(\widehat{f'})_k = \mathrm{i}k\widehat{f}_k.$$

Let $D = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ the differential operator with domain $D(D) = C^1(\mathbb{T})$, and let M be the multiplication operator in $\ell^2(\mathbb{Z})$, $M(a_k) = (ka_k)$, with domain $UD(D)$. Show that

(c) $M = U^{-1}DU$.
(d) D and M are symmetric.
(e) Describe the closure $\overline{M}$, and check that $\overline{M}$ is self-adjoint.
(f) Conclude that D and M are both essentially self-adjoint.