Assignment 6

Problem 1. Let X be the Banach space $C([0,1])$ with the sup norm, and $T : X \to X$ be the operator

$$(Tf)(x) = f(x) + \int_0^x f(y)\mathrm{d}y.$$

Show that $\operatorname{ran} T = X$ and $\ker T = \{0\}$.

Hint: Replace the integral equation by a differential equation, and use appropriate theorems about existence and unicity of solutions.

Problem 2. (Projections)

(a) Let M be a closed subspace of a Hilbert space X, and let P be the orthogonal projector onto M. Show that $P^2 = P$, $\|P\| = 1$ (if $M \neq \{0\}$), and that

$$(Px, y) = (x, Py)$$

for any $x, y \in X$.

(b) Conversely, suppose that $P : X \to X$ is linear, and satisfies $P^2 = P$ and $(Px, y) = (x, Py)$ for all $x, y \in X$. Show that P is the orthogonal projection onto some closed subspace.

Problem 3. Let $M \subset L^2(\mathbb{R})$ be the set of *even* square-summable functions ($f(x) = f(-x)$ for all x). This exercise might be easier to solve in a different order than suggested.

(a) Check that M is a closed subspace.

(b) Let P be the operator defined by $(Pf)(x) = \frac{1}{2}[f(x) + f(-x)]$. Check that P is the orthonal projector onto M.

(c) Find the orthogonal complement $M^\perp$; find an explicit expression for the projector onto $M^\perp$.

Problem 4. (Unitary operators)

(a) Let $U : X_1 \to X_2$ be a unitary operator between Hilbert spaces. Show that $\|U\| = 1$ and $U^* = U^{-1}$.

(b) Let $\mathcal{S}_{\mathbb{N}}$ be the group of permutations (bijections) $\mathbb{N} \to \mathbb{N}$. For $\pi \in \mathcal{S}_{\mathbb{N}}$, define $U_\pi : \ell^2(\mathbb{N}, \mathbb{C}) \to \ell^2(\mathbb{N}, \mathbb{C})$ by

$$U_\pi(x_1, x_2, \dots) = (x_{\pi(1)}, x_{\pi(2)}, \dots).$$

Show that U_π is unitary. Show that this *representation* preserves the group structure of $\mathcal{S}_{\mathbb{N}}$, in the sense

$$U_\pi U_{\pi'} = U_{\pi \circ \pi'}.$$

Problem 5. Let T be a bounded positive operator on a Hilbert space X.

(a) Show that the map $(x, y) \mapsto (x, Ty)$ is an inner product.
(b) Suppose there exists $c > 0$ such that $(x, Tx) \geq c\|x\|^2$ for all $x \in X$. Show that X equipped with the new inner product is a Hilbert space.
(c) Suppose there exists no $c > 0$ such that $(x, Tx) \geq c\|x\|^2$ for all $x \in X$. Show that X equipped with the new inner product is not complete.

This question is not easy, a hint may be necessary. Consider the inclusion map

$$\iota \; : \quad \big(X, (\cdot, \cdot)\big) \to \big(X, (\cdot, T\cdot)\big)$$
$$x \mapsto x,$$

and use the inverse mapping theorem.