

Assignment 5

Problem 1.

- (a) Show that the norm induced by an inner product satisfies the parallelogram identity.
- (b) Let $\|\cdot\|$ be a norm that satisfies the parallelogram identity. Show that the polarization identity defines an inner product. (Hint: You may want to establish the following identity:

$$\|x + y + z\|^2 - \|x - y - z\|^2 = \|x + y\|^2 - \|x - y\|^2 + \|x + z\|^2 - \|x - z\|^2.$$

This may help to prove linearity. As far as I know, this exercise is not easy.)

Problem 2. Show that $L^p(\mathbb{R})$ can be turned into a Hilbert space if and only if $p = 2$, in which case the inner product is

$$(f, g) = \int_{-\infty}^{\infty} \overline{f(x)}g(x)dx.$$

(Hint: Consider two functions with disjoint supports. Then the parallelogram identity reduces to showing that $(a^p + b^p)^{2/p} = a^2 + b^2$ for any positive numbers a, b .)

Problem 3. Let X be a separable infinite-dimensional Hilbert space. Show that:

- (a) Every orthonormal sequence converges weakly to 0.
- (b) The unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \leq 1\}$.

(Note: These properties also hold for nonseparable Hilbert spaces.)

In Problems 4 and 5 we prove that the Fourier functions form an orthonormal basis of $L^2([0, 2\pi])$. Fourier series are then a special case of Proposition 4.7. Problem 6 is a simple, cute, nontrivial, and rather surprising application of Fourier series to the Riemann zeta function.

Problem 4. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denote the circle, where operations $x + y$ and $x - y$ are taken modulo 2π . Recall that the convolution of two functions is defined by

$$(f * g)(x) = \int_{\mathbb{T}} f(x - y)g(y)dy.$$

Let $\varphi_n \geq 0$ be a function on \mathbb{T} that satisfies

$$\int_{\mathbb{T}} \varphi_n(x) dx = 1, \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{|x| > \delta} \varphi_n(x) dx = 0$$

for any $\delta > 0$. Show that, for any $f \in C(\mathbb{T})$, $\varphi_n * f$ converges uniformly to f , i.e. that

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{T}} |(\varphi_n * f)(x) - f(x)| = 0.$$

Problem 5.

- (a) Show that the function $\varphi_n(x) = c_n(1 + \cos x)^n$, where c_n is chosen so that $\int_{\mathbb{T}} \varphi_n = 1$, satisfies the properties of Problem 4.
 (b) Check that $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{T})$.
 (c) Show that $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$. For this, show that

$$\varphi_n(x) = \sum_{k=-n}^n a_{nk} e^{ikx}, \quad \text{with} \quad a_{nk} = 2^{-n} c_n \binom{2n}{n+k}.$$

Then show that for any $f \in C(\mathbb{T})$,

$$(\varphi_n * f)(x) = \sum_{k=-n}^n b_{nk} e^{ikx}, \quad \text{with} \quad b_{nk} = a_{nk} \int_{\mathbb{T}} e^{-ikx} f(y) dy.$$

Then use Problem 4 and Proposition 4.7 of the course to conclude that $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is an orthonormal basis.

Problem 6. Show that $\zeta(2) \equiv \sum_{n \geq 1} n^{-2} = \pi^2/6$. Hint: Use the fact that $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, and that the function $f(x) = x$ has Fourier coefficients proportional to $1/n$.