

## Assignment 4

**Problem 1.** Recall that a map between topological spaces is open iff the image of each open set is open. Show that a linear map between normed spaces is open iff the image of the unit ball (around 0) contains a ball around 0.

**Problem 2.** Let  $X, Y, Z$  be normed spaces, and  $S : X \rightarrow Y$  and  $T : Y \rightarrow Z$  be operators.

- (a) Show that  $\|T \circ S\| \leq \|S\| \|T\|$ .
- (b) Give an example where  $\|T \circ S\| < \|S\| \|T\|$ .

**Problem 3.** Let  $X$  be the space of sequences  $x = (x_1, x_2, \dots)$  of complex numbers with finitely many nonzero terms. We consider the norm  $\|x\| = \sup_i |x_i|$ . Define  $T : X \rightarrow X$  by

$$Tx = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$$

Show that  $T$  is linear and bounded. Show that  $T$  is bijective but that  $T^{-1}$  is unbounded. Does this contradict the Inverse Mapping Theorem?

**Problem 4.** Let  $T : X \rightarrow Y$  be a bounded operator between Banach spaces  $X$  and  $Y$ . Show that, if  $T$  is bijective, there exist constants  $c_1$  and  $c_2$  such that

$$c_1\|x\| \leq \|Tx\| \leq c_2\|x\|$$

for all  $x \in X$ .

**Problem 5.** Here is an exercise that belongs more to analysis than to functional analysis, but it is a beautiful application of Baire Category Theorem. Show that there exist continuous functions on  $[0, 1]$  that are nowhere differentiable.

To that purpose, introduce the set  $A_n$  of functions  $f \in C([0, 1], \mathbb{R})$  such that there exists  $x_0$  (that depends on  $f$ ) such that  $|f(x) - f(x_0)| \leq n|x - x_0|$  for all  $x \in [0, 1]$ .

- (a) Show that  $A_n$  is nowhere dense in  $C([0, 1])$  (with the sup norm).
- (b) Show that if  $f$  is differentiable at some  $x \in [0, 1]$ , then  $f \in \cup_n A_n$ .
- (c) Use Baire Theorem to show that  $\cup_n A_n \subsetneq C([0, 1])$ .