

## Assignment 3

**Problem 1.**

- (a) Use the Hahn-Banach theorem to show that, if  $x_n$  converges weakly to  $x$ , then

$$\liminf_{n \rightarrow \infty} \|x_n\| \geq \|x\|.$$

Hint: Consider a functional  $f$  such that  $\|f\| = 1$  and  $f(x) = \|x\|$ .

- (b) Find examples in  $\ell^p$  and  $L^p$  of weakly convergent sequences, where the inequality above is strict.

**Problem 2.** Let  $X$  be a normed linear space. Show the following:

- (a) If  $\|x_n - x\| \rightarrow 0$ , then  $x_n \rightarrow x$  weakly.  
 (b) If  $X$  is finite-dimensional, then weak convergence is equivalent to norm convergence.  
 (c) If  $X$  is infinite-dimensional, then weak convergence *does not* imply norm convergence. Give an example.  
 (d) If  $x_n$  converges weakly to both  $x$  and  $y$ , then  $x = y$ .

**Problem 3.** Show that  $(\ell^p)^* = \ell^q$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , when  $1 \leq p < \infty$ . More precisely, show that

- (a) any  $a \in \ell^q$  defines a continuous linear functional  $f_a$  on  $\ell^p$  by  $f_a(x) = \sum_n a_n x_n$ ,  $x \in \ell^p$ ;  
 (b) to any functional  $f \in (\ell^p)^*$  there corresponds a sequence  $a \in \ell^q$  such that  $f = f_a$ ;  
 (c) the operator norm of  $f_a$  is equal to the  $\ell^q$  norm of  $a$ .

Are the  $\ell^p$  spaces reflexive?

**Problem 4.** Let  $c \in \ell^\infty$  be the space of convergent sequences, and  $c_0$  be the space of sequences that converge to 0. Show that

- (a)  $c$  and  $c_0$  are Banach spaces;  
 (b)  $(c_0)^* = c^* = \ell^1$ .

Are  $c_0$  and  $c$  reflexive?

**Problem 5.** Find an example of  $1 \leq p, q < \infty$  and a sequence  $(x_n)$  such that  $(x_n)$  converges strongly to 0 in  $\ell^p$ , and weakly (but not strongly) in  $\ell^q$ .