Assignment 1

Problem 1. Here is a counterexample to Proposition 1.2 ($C([a,b])$ is complete w.r.t. $\|\cdot\|_\infty$ norm). Let $f_n(x) = x^n$ on the interval $[0,1]$. The sequence (f_n) converges to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1; \\ 1 & \text{if } x = 1. \end{cases}$$

Then f must be continuous. What is wrong here?

Problem 2. Prove that the Banach space $C([0,1],\mathbb{R})$, with the $\|\cdot\|_\infty$ norm, is separable.

Problem 3. Let $Y \subset \ell^p(\mathbb{C})$ be the set of sequences (of complex numbers) with finitely many non-zero entries, i.e.

$$Y = \big\{(x_1, x_2, \dots) : \exists k < \infty \text{ such that } x_i = 0 \; \forall \, i > k\big\}$$

We equip Y with the usual ℓ^p norm, $1 \leq p \leq \infty$. Explain why Y is not a Banach space. (Prove your assertions!) Show that Y is dense in $\ell^p(\mathbb{C})$.

Problem 4.

(a) Show that $\ell^\infty(\mathbb{C})$ is *not* separable. (*Hint:* This is similar to proving that it is not countable.)

(b) Let $Z \subset \ell^\infty(\mathbb{C})$ be the set of sequences $x = (x_1, x_2, \dots)$ such that $\lim_{k\to\infty} x_k = 0$. Show that Z is a closed subspace.

Problem 5.

(a) Show that $\ell^p(\mathbb{C}) \subsetneq \ell^q(\mathbb{C})$ when $1 \leq p < q \leq \infty$.

(b) Find a sequence of numbers converging to 0, which is *not* in any ℓ^p space with $1 \leq p < \infty$.

(c) If $x \in \ell^p$ for some finite p, show that

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty.$$

Problem 6. Consider two arbitrary norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in $\mathbb{R}^n$. Show that a sequence (x_k) converges with respect to $\|\cdot\|_1$ iff it converges with respect to $\|\cdot\|_2$. Are the limits necessarily the same?

Hint: Show that, if $\|x_k\| \to 0$, then each component of x_k goes to 0. This holds with respect to any norm.