Assignment 9

Problem 1. Prove Proposition 4.14 (a), (b), (c), and (e).

Problem 2. Prove Proposition 4.14 (d), i.e. if (T_n) is a sequence of compact operators that converges to T w.r.t. the operator norm, then T is compact.

Hints:

- A criterion for compactness is that for any bounded sequence (x_m), (Tx_m) contains a convergent subsequence.
- Given a bounded sequence (x_m), use Cantor diagonal process to get a subsequence (y_m) such that $(T_n y_m)$ converges for any fixed n.
- Show that (Ty_m) converges.

Problem 3. Let X be a separable Hilbert space. An operator $T : X \to X$ is *Hilbert-Schmidt* if there exists an orthonormal basis (e_n) such that $\sum_n \|Te_n\|^2 < \infty$.

(a) Show that Hilbert-Schmidt operators are compact (hence bounded).

We define the norm of a Hilbert-Schmidt operator T by

$$\|T\|_{\mathrm{HS}} = \Big(\sum_{n\geq 1} \|Te_n\|^2\Big)^{1/2}.$$

(b) Show that $\|\cdot\|_{\mathrm{HS}}$ is a norm.

(c) Show that the Hilbert-Schmidt norm does not depend on the choice of the orthonormal basis.

Problem 4. Let T be the multiplication operator by a function g. That is, we define $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$Tf(x) = g(x)f(x),$$

where g is a fixed function, that we suppose to be continuous and bounded. Prove that

(a) T is bounded;

(b) the spectrum is $\sigma(T) = \overline{\{g(x) : x \in \mathbb{R}\}}$;

(c) give an example of a continuous, bounded function g such that T has eigenvalues;

(d) can you find a function $g \not\equiv 0$ such that T is compact?

Problem 5. Let $K : L^2([0,1]) \to L^2([0,1)]$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)\mathrm{d}y.$$

(a) Find the adjoint operator K^*.

(b) Show that $\|K\| = \frac{2}{\pi}$.

(c) Show that $0 \in \sigma_c(K)$.

(d) The lottery question. Give a correct solution to Michael Doré by Tuesday, and enter the lottery for a bottle wine! Show that $\sigma(K) = \sigma_c(K) = \{0\}$.

For (d), you may use the notion of *spectral radius*. The spectral radius of a bounded operator T is defined by $r(T) = \sup_{\alpha \in \sigma(T)} |\alpha|$. Prove that it satisfies

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

Then show that $r(K) = 0$.