Assignment 8

Problem 1. Let X be a Hilbert space. Show that if T is a bounded, positive definite operator on X, then $\big(X, (\cdot, T\cdot)\big)$ is a Hilbert space iff there exists $c > 0$ such that

$$(x, Tx) \geq c\|x\|^2$$

for any x. Hint: One direction is easy. For the other direction, consider the inclusion map

$$\begin{aligned} \iota \ : \quad \big(X, (\cdot, \cdot)\big) &\to \big(X, (\cdot, T\cdot)\big) \\ x &\mapsto x, \end{aligned}$$

and use the inverse mapping theorem. (Thanks to Michael Doré for the hint!)

Problem 2. Let $T \in \mathcal{B}(X)$, and $\alpha, \beta \in \rho(T)$. Let $R_\alpha = (T - \alpha\mathbb{1})^{-1}$ denote the resolvent.

(a) Show that it satisfies the *Hilbert relation* (or *resolvent equation*)

$$R_\alpha - R_\beta = (\alpha - \beta) R_\alpha R_\beta.$$

(b) Show that $R_\alpha R_\beta = R_\beta R_\alpha$.

Problem 3. (Shift operators) We consider the right and left shift operators on $\ell^2(\mathbb{N})$:

$$\begin{aligned} S(x_1, x_2, \dots) &= (0, x_1, x_2, \dots), \\ T(x_1, x_2, \dots) &= (x_2, x_3, \dots). \end{aligned}$$

(a) Find $\|S\|$, $\|T\|$, S^*, T^*, S^{-1}, T^{-1}.

(b) Find $\operatorname{ran} S$, $\operatorname{ran} T$, $\ker S$, $\ker T$, and check that

$$\operatorname{ran} S = (\ker T)^\perp, \qquad \operatorname{ran} T = (\ker S)^\perp.$$

(c) Find the spectrum of S and T.

Problem 4. Let $T \in \mathcal{B}(X)$. Show that

(a) If $u_1, \dots, u_n \in X$ are eigenvectors of T corresponding to distinct eigenvalues, then $\{u_1, \dots, u_n\}$ forms a linearly independent set.

(b) If $T = T^*$, and M is an invariant subspace (that is, $T(M) \subset M$), then $M^\perp$ is also invariant.

Problem 5. The lottery question. Give a correct solution to Michael Doré by Tuesday, and enter the lottery for a bottle wine!

Let ℓ_0 be the space of all sequences of complex numbers $(x_1, x_2, \dots)$ with finitely many nonzero entries. Can you find a norm such that ℓ_0 is complete? If yes, give it. If not, prove there exists none.

(I heard that Baire category theorem might help.)