Assignment 6

Problem 1. Let X be a separable infinite-dimensional Hilbert space. Show that:

(a) Every orthonormal sequence converges weakly to 0.

(b) The unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \leq 1\}$.

(Note: These properties also hold for nonseparable Hilbert spaces.)

Problem 2. Let X and Y be two Hilbert spaces. Define the direct sum of X and Y by

$$X \oplus Y = \big\{(x, y) : x \in X, y \in Y\big\},$$

with the inner product

$$\big((x, y), (x', y')\big)_{X \oplus Y} = (x, x')_X + (y, y')_Y.$$

Show that $X \oplus Y$ is a Hilbert space. Find the orthogonal complement of the subspace $\{(x, 0) : x \in X\}$.

Problem 3. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denote the circle, where operations $x + y$ and $x - y$ are taken modulo 2π. Recall that the convolution of two functions is defined by

$$(f * g)(x) = \int_{\mathbb{T}} f(x - y) g(y) \mathrm{d}y.$$

Let $\varphi_n \geq 0$ be a function on $\mathbb{T}$ that satisfies

$$\int_{\mathbb{T}} \varphi_n(x) \mathrm{d}x = 1, \quad \text{and} \quad \lim_{n \to \infty} \int_{|x| > \delta} \varphi_n(x) \mathrm{d}x = 0$$

for any $\delta > 0$. Show that, for any $f \in C(\mathbb{T})$, $\varphi_n * f$ converges uniformly to f, i.e. that

$$\lim_{n \to \infty} \sup_{x \in \mathbb{T}} \big|(\varphi_n * f)(x) - f(x)\big| = 0.$$

Problem 4.

(a) Show that the function $\varphi_n(x) = c_n(1 + \cos x)^n$, where c_n is chosen so that $\int_{\mathbb{T}} \varphi_n = 1$, satisfies the properties of Problem 3.

(b) Check that $\{\, \mathrm{e}^{\mathrm{i}kx} \,\}_{k \in \mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{T})$.

(c) Show that $\{ \mathrm{e}^{\mathrm{i}kx} \}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$. For this, show that

$$\varphi_n(x) = \sum_{k=-n}^{n} a_{nk}\, \mathrm{e}^{\mathrm{i}kx}\,, \qquad \text{with} \qquad a_{nk} = 2^{-n} c_n \tbinom{2n}{n+k}\,.$$

Then show that for any $f \in C(\mathbb{T})$,

$$(\varphi_n * f)(x) = \sum_{k=-n}^{n} b_k\, \mathrm{e}^{\mathrm{i}kx}\,, \qquad \text{with} \qquad b_k = a_{nk} \int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}kx} f(y)\mathrm{d}y.$$

Then use Problem 3 and Proposition 3.7 of the course to conclude that $\{ \mathrm{e}^{\mathrm{i}kx} \}_{k\in\mathbb{Z}}$ is an orthonormal basis.

Problem 5. Show that $\zeta(2) \equiv \sum_{n\geq 1} n^{-2} = \pi^2/6$. Hint: Use the fact that $\{ \mathrm{e}^{\mathrm{i}kx} \}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, and that the function $f(x) = x$ has Fourier coefficients proportional to $1/n$.