Assignment 5

Problem 1. Show that the induced norm $\|x\| = \sqrt{(x,x)}$ satisfies the axioms for a norm.

Problem 2.

(a) Show that the norm induced by an inner product satisfies the parallelogram identity.

(b) Let $\|\cdot\|$ be a norm that satisfies the parallelogram identity. Show that the polarization identity defines an inner product. (Hint: You may want to establish the following identity:

$$\|x+y+z\|^2 - \|x-y-z\|^2 = \|x+y\|^2 - \|x-y\|^2 + \|x+z\|^2 - \|x-z\|^2.$$

This may help to prove linearity. As far as I know, this exercise is not easy.)

Problem 3. Prove Proposition 3.2. (Hint: for (b), use the projection theorem.)

Problem 4. Let M be a (nonempty) closed subspace of a Hilbert space X, and let P be the orthogonal projection on M. Prove that P is a bounded linear map (and find its norm!), and that $P^2 = P$.

Problem 5. Show that $L^p(\mathbb{R})$ can be turned into a Hilbert space if and only if $p = 2$, in which case the inner product is

$$(f,g) = \int_{-\infty}^{\infty} \overline{f(x)} g(x) \mathrm{d}x.$$

(Hint: Consider two functions with disjoint supports. Then the parallelogram identity reduces to showing that $(a^p + b^p)^{2/p} = a^2 + b^2$ for any positive numbers a, b.)