Assignment 4

Problem 1. Recall that a map between topological spaces is open iff the image of each open set is open. Show that a linear map between normed spaces is open iff the image of the unit ball (around 0) contains a ball around 0.

Problem 2. Use the Hahn-Banach theorem to show that, if x_n converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n\| \geq \|x\|.$$

Hint: Consider a functional f such that $\|f\| = 1$ and $f(x) = \|x\|$.

Problem 3. Let X be the space of sequences $x = (x_1, x_2, \dots)$ of complex numbers with finitely many nonzero terms. We consider the norm $\|x\| = \sup_i |x_i|$. Define $T : X \to X$ by

$$Tx = (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots)$$

Show that T is linear and bounded. Show that T is bijective but that T^{-1} is unbounded. Does this contradict the Inverse Mapping Theorem?

Problem 4. Let $T : X \to Y$ be a bounded operator between Banach spaces X and Y. Show that, if T is bijective, there exist constants c_1 and c_2 such that

$$c_1\|x\| \leq \|Tx\| \leq c_2\|x\|$$

for all $x \in X$.

Problem 5. Here is an exercise that belongs more to analysis than real analysis, but it is a beautiful application of Baire Category Theorem. Show that there exist continuous functions on $[0, 1]$ that are nowhere differentiable.

To that purpose, introduce the set A_n of functions $f \in C([0, 1], \mathbb{R})$ such that there exists x_0 (that depends on f) such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0, 1]$.

(a) Show that E_n is nowhere dense in $C([0, 1])$ (with the sup norm).
(b) Show that if f is differentiable at some $x \in [0, 1]$, then $f \in \cup_n A_n$.
(c) Use Baire Theorem to show that $\cup_n A_n \subsetneq C([0, 1])$.