Assignment 3

Problem 1. Consider the functional $\delta : C([0,1]) \to \mathbb{R}$ such that $\delta(f) = f(0)$. Find the norm of δ, when $C([0,1])$ is equipped with the L^p norm, $1 \le p \le \infty$.

Problem 2. Let X be a normed linear space. Show the following:

(a) If $\|x_n - x\| \to 0$, then $x_n \to x$ weakly.
(b) If X is finite-dimensional, then weak convergence is equivalent to norm convergence.
(c) If X is infinite-dimensional, then weak convergence *does not* imply norm convergence. Give an example.
(d) If x_n converges weakly to both x and y, then $x = y$.

Problem 3. Show that $(\ell^p)^* = \ell^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, when $1 \le p < \infty$. More precisely, show that

(i) any $a \in \ell^q$ defines a continuous linear functional f_a on ℓ^p by $f_a(x) = \sum_n a_n x_n$, $x \in \ell^p$;
(ii) to any functional $f \in (\ell^p)^*$ there corresponds a sequence $a \in \ell^q$ such that $f = f_a$;
(iii) the operator norm of f_a is equal to the ℓ^q norm of a.

Are the ℓ^p spaces reflexive?

Problem 4. Let $c \in \ell^\infty$ be the space of convergent sequences, and c_0 be the space of sequences that converge to 0. Show that

(i) c and c_0 are Banach spaces;
(ii) $(c_0)^* = c^* = \ell^1$.

Are c_0 and c reflexive?

Problem 5. Prove that ℓ^∞ is not separable.