# The Critical Temperature of Dilute Bose Gases

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# Program

1. Bose-Einstein condensation
2. Effects of interactions on critical temperature
3. The density bound
4. Feynman-Kac representation
5. Estimates of integral kernels

# Quantum many-body system

$N$ particles in box $\Lambda \subset \mathbb{R}^d$
State space is Hilbert space $\otimes_{i=1}^{N} L^2(\Lambda) \simeq L^2(\Lambda^N)$

Hamiltonian given by **Schrödinger operator**

$$H = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$$

with $\Delta_i = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial x_{i,\nu}^2}$ the Laplacian and $U \geqslant 0$ a function with compact support. (Periodic boundary conditions)

System of identical **bosons**: state space is $L^2_{\text{sym}}(\Lambda^N)$ , the space of $L^2$ functions that are symmetric with respect to $N$ arguments

# Bose-Einstein condensation

Understood by Einstein in 1924 for the **ideal** gas (no interactions)
Fourier space: $\Lambda_* = \frac{1}{L}\mathbb{Z}^d$

$$\ell^2_{\text{sym}}(\Lambda_*^N) \;\simeq\; \text{span}\Big\{ \boldsymbol{n} \in \mathbb{N}^{\Lambda_*} : \sum_{k\in\Lambda_*} n_k = N \Big\}$$

The space spanned by **occupation numbers**.
Hamiltonian is now a multiplication operator:

$$H|\boldsymbol{n}\rangle = \sum_{k\in\Lambda_*} 4\pi^2 k^2 n_k |\boldsymbol{n}\rangle$$

# Bose-Einstein condensation

Statistical mechanics: Expectation of observables (i.e. self-adjoint operators on state space) is defined by

$$\langle A \rangle = \frac{\operatorname{Tr} A \, \mathrm{e}^{-\beta H}}{\operatorname{Tr} \, \mathrm{e}^{-\beta H}}$$

where $\beta = 1/T$ is inverse temperature
Let $\rho = N/|\Lambda|$ the particle density, and consider the observable $A = \frac{n_0}{L^3}$ with $n_0 = n_{k=0}$

# Bose-Einstein condensation

**Bose-Einstein condensation** ($d = 3$ ): There is a critical density, $\rho_c = \frac{\zeta(3/2)}{(4\pi\beta)^{3/2}}$ (with $\zeta(3/2) = \sum n^{-3/2} = 2.612...$ ) such that

$$\lim_{\substack{L,N\to\infty \\ N/L^3=\rho}} \frac{\operatorname{Tr} \frac{n_0}{L^3} \, \mathrm{e}^{-\beta H}}{\operatorname{Tr} \, \mathrm{e}^{-\beta H}} = \begin{cases} 0 & \text{if } \rho \leqslant \rho_c \\ \rho - \rho_c & \text{if } \rho \geqslant \rho_c \end{cases}$$

Alternatively, there is a critical temperature $T_c = 4\pi(\rho/\zeta(\frac{3}{2}))^{2/3}$

Physical manifestations of BEC: superfluidity, superconductivity. Also present in optics and in turbulence

The **scattering length** $a$ is a number that characterizes the potential function for dilute quantum systems. Hard core potentials: $a$ is the radius of the hard core. For integrable potentials, $a_0 = \frac{1}{8\pi}\int U(x)\mathrm{d}x$ is the first Born approximation to the scattering length. Otherwise, let $R$ be larger than the radius $R_0$ of $U$ , and consider the energy functional

$$\mathcal{E}_R(\psi) = \int_{B_R} \Big( |\nabla\psi(x)|^2 + U(x)|\psi(x)|^2 \Big) \mathrm{d}x$$

with $B_R$ the ball of radius $R$ in $\mathbb{R}^3$ . Let $\psi_0$ be the minimizer of $\mathcal{E}_R$ over $H^1$ functions that satisfy the boundary condition $\psi_0(x) = 1$ when $|x| = R$ . $\psi_0$ is spherically symmetric and it satisfies the equation

$$-\Delta\psi_0(x) + U(x)\psi_0(x) = 0$$

For $R_0 \leqslant |x| \leqslant R$ , we have

$$\psi_0(x) = \frac{1 - a/|x|}{1 - a/R}$$

for a number $a$ , the **scattering length**. Cf Lieb, Seiringer, Solovej, Yngvason, *The Mathematics of the Bose Gas and its Condensation*

# Effects of particle interactions on critical temperature

$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{i<j} U(x_i - x_j)$ , $U(x) \geqslant 0$ with scattering length $a$

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2000 Reppy et. al.: $c = 5.1$

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A partial but rigorous result:

**Theorem** (with R. Seiringer, 2009)

*There is no BEC when* $\dfrac{\Delta T_c}{T_c} > 5.09\sqrt{a\rho^{1/3}}$

# Grand-canonical ensemble

(Laplace transform with respect to $N$ )

$$\langle A \rangle = \frac{1}{Z} \sum_{N \geqslant 0} z^N \mathrm{Tr}_{L^2_{\mathrm{sym}}(\Lambda^N)} A \, \mathrm{e}^{-\beta H}$$

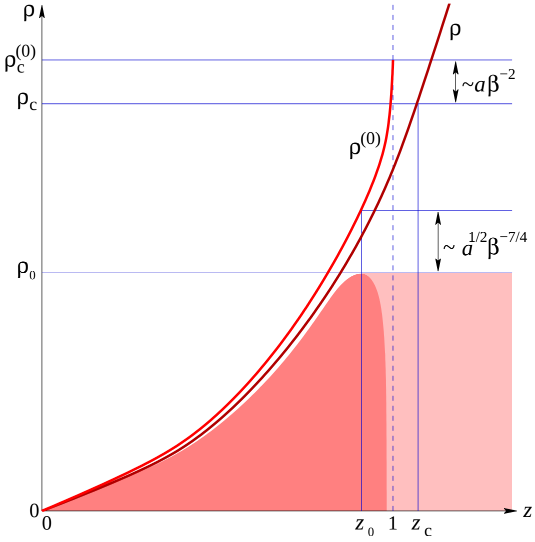
Parameters are $\beta, \Lambda, z$ ($z$ : fugacity)
The density is given by $\rho(z) = \frac{\langle N \rangle}{|\Lambda|}$ ; $\rho(z)$ is increasing in $z$
FACT: There is no Bose-Einstein condensation when $z < 1$ . This holds for any repulsive interaction (see Bratteli-Robinson)

We need information on $\rho(z)$
Known: $\rho^{(0)}(z) = (4\pi\beta)^{-3/2} \sum_{n \geqslant 1} z^n / n^{3/2}$

**Theorem** (with R. Seiringer, 2009)

$$\rho(z) \geqslant \rho^{(0)}(z) - C \frac{a}{\beta^2 \sqrt{-\log z}}$$

# Illustration



# Feynman-Kac representation for the Bose gas

◇ Allows to get suitable bounds, by dropping certain terms

Let $W_{xy}^t$ be the Wiener measure for the Brownian bridge between $x$ and $y$ in time $t$ . Let $\boldsymbol{\omega} = (x, k, \omega)$ stand for a "winding path" from $x$ to $x$ in time $2\beta k$ , and

$$V(\boldsymbol{\omega}) = \frac{1}{2} \sum_{0 \leqslant \ell < m \leqslant k-1} \int_0^{2\beta} U\big(\omega(2\ell\beta + s) - \omega(2m\beta + s)\big) \mathrm{d}s$$

Define the measure $\mu$ by

$$\int \mathrm{d}\mu(\boldsymbol{\omega}) \dots \equiv \sum_{k \geqslant 1} \frac{z^k}{k} \int_\Lambda \mathrm{d}x \int \mathrm{d}W_{xx}^{2\beta k}(\omega)\, \mathrm{e}^{-V(\boldsymbol{\omega})} \dots$$

# Feynman-Kac representation for the Bose gas

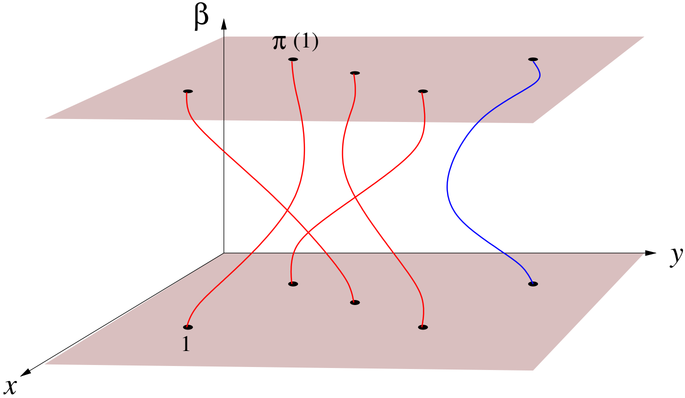
The partition function $Z = \sum_{N \geqslant 0} z^N \mathrm{Tr}\, \mathrm{e}^{-\beta H}$ is then given by

$$Z = \sum_{n \geqslant 0} \frac{1}{n!} \int \mathrm{d}\mu(\boldsymbol{\omega}_1) \dots \mathrm{d}\mu(\boldsymbol{\omega}_n)\, \mathrm{e}^{-\sum_{1 \leqslant i<j \leqslant n} V(\boldsymbol{\omega}_i, \boldsymbol{\omega}_j)}$$

where the two-path interaction is

$$V(\boldsymbol{\omega}, \boldsymbol{\omega}') = \frac{1}{2} \sum_{\ell=0}^{k-1} \sum_{\ell'=0}^{k'-1} \int_0^{2\beta} U\big(\omega(2\beta\ell + s) - \omega'(2\beta\ell' + s)\big) \mathrm{d}s$$

# Space-time picture



# Feynman-Kac representation for the Bose gas

Similarly, the density $\rho(z)$ is given by

$$\rho(z) = \frac{1}{|\Lambda| Z} \sum_{n \geqslant 1} \frac{1}{(n-1)!} \int \mathrm{d}\mu(\boldsymbol{\omega}_1) k_1 \int \mathrm{d}\mu(\boldsymbol{\omega}_2) \dots \int \mathrm{d}\mu(\boldsymbol{\omega}_n) \, \mathrm{e}^{-\sum_{1 \leqslant i<j \leqslant n} V(\boldsymbol{\omega}_i, \boldsymbol{\omega}_j)}$$

Since $V \geqslant 0$ , one immediately gets the **upper bound**

$$\rho(z) \;\leqslant\; \frac{1}{|\Lambda|} \int \mathrm{d}\mu(\boldsymbol{\omega}_1) k_1 \;\leqslant\; \rho^{(0)}(z)$$

For the **lower bound**, one uses $\mathrm{e}^{-\sum_i V_i} \geqslant 1 - \sum_i (1 - \mathrm{e}^{-V_i})$ in various ways, to obtain

$$\rho(z) \geqslant \rho^{(0)}(z) - \frac{C}{\beta^3 \sqrt{-\log z}} \int K(x,y) \mathrm{d}x \mathrm{d}y$$

where $K(x,y) \equiv \int \bigl[1 - \mathrm{e}^{-\frac{1}{4} \int_0^{4\beta} U(\omega(s)) \mathrm{d}s}\bigr] \mathrm{d}W_{xy}^{4\beta}(\omega)$ is the integral kernel of $\mathrm{e}^{2\beta\Delta} - \mathrm{e}^{\beta(2\Delta - U)}$ (Feynman-Kac formula)

# Scattering estimates

It remains to study the quantity

$$a(\beta) \equiv \frac{1}{8\pi\beta} \int K(x,y)\mathrm{d}x\mathrm{d}y$$

Using $1 - \mathrm{e}^{-u} \leqslant u$ , we have

$$\begin{aligned}\int K(x,y)\mathrm{d}x\mathrm{d}y &\leqslant \frac{1}{4}\int_0^{4\beta} \mathrm{d}s \int \mathrm{d}y \int \mathrm{d}W_{0y}^{4\beta}(\omega) \int \mathrm{d}x\, U\big(\omega(s)+x\big) \\ &= \beta \int U(z)\mathrm{d}z\end{aligned}$$

In scattering theory, $a_0 = \frac{1}{8\pi}\int U(z)\mathrm{d}z$ is called the **first Born approximation** of the scattering length

Then $a(\beta) \leqslant a_0$ . OK if potential $U$ is integrable and small in a suitable sense. One needs another method for more general potentials

# Variational principle

$$a(\beta) = \frac{1}{8\pi} \inf_{\psi \in H^1(\mathbb{R}^d)} \Big[ \int_{\mathbb{R}^d} \big( 2|\nabla\psi(x)|^2 + U(x)|1 - \psi(x)|^2 \big) \mathrm{d}x + \frac{1}{\beta} \langle \psi | f(\beta(-2\Delta + U)) | \psi \rangle \Big]$$

where $f$ is the decreasing function

$$f(t) = t \frac{1 - \mathrm{e}^{-t}}{t - 1 + \mathrm{e}^{-t}}$$

Notice that $1 \leqslant f \leqslant 2$ . The variational principle shows that $a(\beta)$ is decreasing in $\beta$ and that

$$\lim_{\beta \to 0} a(\beta) = a_0, \quad \lim_{\beta \to \infty} a(\beta) = a$$

Then $a \leqslant a(\beta) \leqslant a_0$ . One can also show that

$$a(\beta) = a\big(1 + O(\sqrt{a\beta^{-1/2}})\big)$$

# Two more scattering estimates

We also need to estimate the following expressions:

$$a'(\beta) = (8\pi\beta)^{1/2} \int K(x,x) \mathrm{d}x$$

$$a''(\beta) = (8\pi\beta)^{1/2} \int K(x,-x) \mathrm{d}x$$

**Lemma**

$$\max\{a'(\beta), a''(\beta)\} \leqslant 2^{d/2} a(\beta/2)$$

For $d = 3$ , we can also show that

$$\lim_{\beta\to\infty} a'(\beta) = a, \quad \lim_{\beta\to\infty} a''(\beta) = a$$

We do not control $a'$ and $a''$ as well as we control $a$ , but this is not important for our purpose

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THANK YOU!